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MODULES WHOSE ENDOMORPHISM RINGS HAVE ISOMORPHIC MAXIMAL LEFT AND RIGHT QUOTIENT RINGS

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ABSTRACT. Let ${}_RM$ be a left R-module such that $\mathrm{Hom}_R(M, U) \neq 0$ for any nonzero submodule U of M, let $E(M)$ denote the injective hull of M, and let B (resp. A) denote the ring of R-endomorphisms of M (resp. $E(M)$). It is known that if M is nonsingular then B is left nonsingular and A is the maximal left quotient ring of B. We give here necessary and sufficient conditions on M for B to be right nonsingular and for A to be the maximal right quotient ring of B.

1. Introduction and preliminaries. In [5], Utumi gave the solution of the following problem: given a ring S which is both left and right nonsingular, when is the maximal left quotient ring (MLQR) of S isomorphic to its maximal right quotient ring (MRQR)? He proved that this holds if and only if the converses of the nonsingular properties hold in S, namely, if and only if every left ideal of S, which has zero right annihilator, is essential in S and every right ideal of S, which has zero left annihilator, is essential in S [**5**, Theorem 3.3]. We consider here analogous questions for the endomorphism ring of an R-module.

Let ${}_RM$ be a left R-module, where R is a ring with 1, and $B = \mathrm{End}_R(M)$ its ring of R-endomorphisms. The following notation will be used in the sequel: If U is a submodule of M, then

$$I_B(U) = \{b \in B : Mb \subset U\}, \qquad r_B(U) = \{b \in B : Ub = 0\},$$
$$l_R(U) = \{r \in R : rU = 0\}.$$

If J is a right ideal of B, then $l_M(J) = \{m \in M : mJ = 0\}$. $X \subset' Y$ means that X is an essential submodule of Y, i.e. X intersects nontrivially every nonzero submodule of Y; in case I is a left, right or two-sided ideal of a ring S, then ${}_sI \subset' {}_sS$ (resp. $I_s \subset' S_s$) will indicate that I is essential in S as a left (resp. right) ideal of S.

Recalling that ${}_RM$ is said to be *nonsingular* in case the only submodule of M with essential (left) annihilator in R is the zero submodule—in our notation: $l_R(U) \subset' {}_RR \Rightarrow U = 0$—we will call M *cononsingular* in case the only submodule of M with essential (right) annihilator in B is the zero submodule—i.e. $r_B(U) \subset' B_B \Rightarrow U = 0$. Let $E(M)$ denote the injective hull of M and $A = \mathrm{End}_R[E(M)]$ its ring of R-endomorphisms. It is known that if M is nonsingular then A is a (von Neumann) regular left, self-injective ring. If we impose a mild nondegeneracy

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condition on M, namely assuming M is *retractable*, i.e. $I_B(U) \neq 0$ for any nonzero submodule U of M, then, when ${}_RM$ is nonsingular, B is left nonsingular and A is the MLQR of B [4]. It is natural to ask here: what properties of M will make B also right nonsingular and what properties of M will make A isomorphic to the MRQR of B?

In answer to the first question, we show in Proposition 1 that, for a retractable nonsingular ${}_RM$, B is right nonsingular if and only if M is cononsingular. As for the second question, it turns out that the required conditions on M closely parallel the conditions of Utumi mentioned above; specifically, we show in Theorem 2 that B has isomorphic MLQR and MRQR if and only if

(a) any (left) submodule of M with zero (right) annihilator in B is essential in M—i.e. $r_B(U) = 0 \Rightarrow U \subset' M$; and

(b) any right ideal of B with zero (left) annihilator in M is essential in B—i.e. $l_M(J) = 0 \Rightarrow J_B \subset' B_B$.

We note that property (a) is the converse of the following well-known property of a nonsingular module: if ${}_RM$ is nonsingular then any essential submodule of M has zero annihilator in B—i.e.: $U \subset' M \Rightarrow r_B(U) = 0$; while property (b) is the converse of a corresponding property of cononsingular modules: if M is cononsingular then any essential right ideal of B has zero kernel—i.e. $J_B \subset' B_B \Rightarrow l_M(J) = 0$. Examples of modules satisfying the various conditions are given in the last paragraph.

2. Endomorphism rings with isomorphic left and right quotient rings. Henceforth, unless otherwise indicated, let ${}_RM$ be a retractable, nonsingular left R-module, so that, in particular, B is left nonsingular and A is the MLQR of B.

PROPOSITION 1. *B is right nonsingular if and only if M is cononsingular.*

PROOF. Assume that B is right nonsingular, so that any element of B with essential right annihilator must be zero. If U is any submodule of M, then clearly $I_B(U)r_B(U) = 0$. Hence, given the right nonsingularity of B, if $r_B(U)$ is essential in B, then $I_B(U) = 0$, which implies, by retractability of M, that $U = 0$. Therefore M is cononsingular.

Conversely, assume that M is cononsingular, and suppose that $bJ = 0$ for some b in B, with J an essential right ideal of B. Then $MbJ = 0$ implies that Mb is contained in $l_M(J)$; hence $r_B(Mb)$ contains $r_Bl_M(J)$ which contains J. Since J is essential in B, this implies that $r_B(Mb)$ is essential in B, hence, by cononsingularity of M, $Mb = 0$, i.e. $b = 0$ and B is right nonsingular.

A left nonsingular ring S, i.e. one in which every essential left ideal has zero right annihilator, is called a *left Utumi ring* in case any left ideal of S with zero right annihilator in S is essential in S. In a nonsingular left R-module ${}_RM$, an essential (left) submodule, U, has zero (right) annihilator in $B - U \subset' M \Rightarrow r_B(U) = 0$. We will call a nonsingular ${}_RM$ a *Utumi module* in case the converse of this property holds in M, i.e. in case any submodule of M with zero annihilator in B is essential in $M - r_B(U) = 0 \Rightarrow U \subset' M$.

The definition of a *right Utumi ring* is the right-left symmetry of the definition of a left Utumi ring. Using this terminology, Utumi's theorem may be restated as follows: A right and left nonsingular ring S has isomorphic MLQR and MRQR if and only if S is both right and left Utumi. If ${}_RM$ is cononsingular, it follows easily that any essential right ideal of B has zero kernel in M — $J_B \subset' B_B \Rightarrow l_M(J) = 0$; let us call a cononsingular ${}_RM$ a *co-Utumi module* in case the converse of this property holds in M, i.e. in case any right ideal of B with zero annihilator in M is essential in B — $l_M(J) = 0 \Rightarrow J_B \subset' B_B$.

Our main result may now be stated as follows:

THEOREM 2. *If M is a retractable, nonsingular, cononsingular left R-module, then $A = \operatorname{End}_R(E(M))$ is both the MLQR and the MRQR of $B = \operatorname{End}_R(M)$ if and only if M is both a Utumi and a co-Utumi module.*

Theorem 2 follows immediately from the following two lemmas.

LEMMA 3. *B is a left Utumi ring if and only if M is a Utumi module.*

PROOF. It is shown in [**5**] that the left nonsingular ring B is a left Utumi ring if and only if B has nonzero intersection with every nonzero right ideal of its MLQR, A [**5**, Theorem 2.2]. But, by Theorem 3.5 of [**3**] ((iv)⇒(iii)), since $A = \operatorname{End}_R(E(M))$, this holds if and only if $r_B(U) = 0$ for every U which is not essential in M, i.e. if and only if M is a Utumi module.

LEMMA 4. *If M is cononsingular, then B is a right Utumi ring if and only if M is a co-Utumi module.*

PROOF. Assume that B is right Utumi and suppose that $l_M(J) = 0$ for some right ideal, J, of B. Then $I_B l_M(J) = 0$. But $I_B l_M(J)$ is equal to the left annihilator, $\mathcal{L}(J)$, of J in B. Hence, the right Utumi property implies that J is essential in B; so M is co-Utumi.

Conversely, assume that M is co-Utumi and let J be a right ideal of B with zero left annihilator, $\mathcal{L}(J)$, in B. Then $I_B l_M(J) = \mathcal{L}(J) = 0$ implies, since M is retractable, that $l_M(J) = 0$, which implies that J is essential in B since M is co-Utumi. Hence B is right Utumi.

EXAMPLES. Any free module, in fact any generator, is retractable, as is any semisimple module, any torsionless module over a semiprime ring and any ${}_RM$ such that $(\operatorname{Trace}_R M)m \neq 0$ whenever $0 \neq m \in M$.

Examples of Utumi modules are CS-modules, i.e. modules in which every complement submodule is a direct summand (see e.g. [**1** and **2**] for examples and properties of such modules). To see that a CS-module is Utumi, let ${}_RM$ be any CS-module and let U be any submodule of M such that $r_B(U) = 0$. Since the essential-closure, U^e, of U is a direct summand in M, there is a submodule, V of M such that $M = U^e \oplus V$, and a $b \in B$ such that $U^e b = 0$ and $vb = v$ for $v \in V$. Then $r_B(U) = 0$ implies that $b = 0$ and so $V = 0$ and U is essential in M.

If M is a retractable, nonsingular CS-module, then M is also cononsingular, for, by Corollary 3.6 of [**3**], since M is retractable and Utumi, B is Baer if and only if every essentially-closed submodule of M is a direct summand in M. Thus, since M is CS, B is Baer, hence, in particular, right nonsingular, which, by Proposition 1, implies that M is cononsingular.

Finally, an example of a module which is both Utumi and co-Utumi is obtained when M is taken to be a finite-dimensional (in the sense of Goldie) torsionless module over a ring R which possesses a semisimple two-sided quotient ring, for then A is a semisimple two-sided quotient ring of B [**6**, Theorems 2.3 and 3.3].

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