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NOMOGRAPHIC FUNCTIONS ARE NOWHERE DENSE

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ABSTRACT. A function f of n variables is nomographic if it can be represented in the format

$$f(x_1,\dots,x_n) = h(\phi_1(x_1) + \cdots + \phi_n(x_n))$$

where the ϕ_i and h are continuous. Every continuous function of n variables has a representation as a sum of not more than $2n+1$ nomographic functions [9]. This paper gives a constructive proof that the nomographic functions form a nowhere dense subset of the space $C[I^n]$.

Let $I = [-1, 1]$ and $I^k = I \times I \times \cdots \times I$. The class $\mathcal{N}^k$ of nomographic functions of k variables are those that on I^k have a representation in the special format

$$f(x_1, x_2, \dots, x_k) = h(\phi_1(x_1) + \phi_2(x_2) + \cdots + \phi_k(x_k)) \tag{1}$$

where the ϕ_k and h are real-valued continuous functions on $-\infty < t < \infty$. Interest in $\mathcal{N}^k$ revived when Kolmogorov used these functions in 1957 to settle Hilbert's 13th problem by showing that every continuous function on I^k could be written as the sum of $2k+1$ functions from $\mathcal{N}^k$. (See [**9, 10**].)

Formula (1) makes it natural to conjecture that $\mathcal{N}^k$ is a sparse subset of the space $C[I^k]$ of all real continuous functions on I^k, with the usual norm $\|g\| = \max|g(p)|$, as suggested in [**2**]. This does not conflict with the Kolmogorov result; for example, $[0,1]$ is the algebraic sum of two copies of a nowhere dense subset E. That $\mathcal{N}^k$ should be sparse is more evident when smoothness is required. If $f \in \mathcal{N}^2$ has component functions h, ϕ_1, and ϕ_2 which are in C''', then f must satisfy a third order PDE that is characteristic for $\mathcal{N}^2$. (See [**4**].) In addition, nowhere denseness is known to follow smoothness in a number of other studies of superposition classes. (See [**6, 7, 8**].) In the present paper, we present a direct constructive proof that $\mathcal{N}^k$ is nowhere dense in $C[I^k]$, with no differentiability restrictions on h or ϕ_j; indeed, we do not require that h be continuous. We remark that $\mathcal{N}^2$ is not uniformly closed. (See [**1** or **3**].)

We prove a more general theorem, and then verify later that the requisite property is shared by the class $\mathcal{N}^k$.

Let D be a compact set in R^n with nonvoid interior, and $C[D]$ be the Banach space of real-valued continuous functions on D, with the uniform convergence norm $\|F\|_D = \max_{p\in D}|F(p)|$. Let $\mathcal{F}$ be a subset of $C[D]$ which we will prove is nowhere dense. The key requirement we need is the existence of special functions in $C[D]$ that fail to belong locally to the closure of $\mathcal{F}$.

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THEOREM 1. *Suppose that there is a point p_0 interior to D such that for any real c, there exists $g \in C[D]$ with $g(p_0) = c$ such that for every compact neighborhood V of p_0*

$$\inf_{f \in \mathcal{F}} \|f - g\|_V > 0. \tag{2}$$

Then, $\mathcal{F}$ is nowhere dense in $C[D]$.

PROOF. Let U be any nonempty open set in $C[D]$; we will produce another nonempty open set $U_1 \subset U$ disjoint from $\mathcal{F}$. Choose $G_0 \in U$ and $r > 0$ such that $\|G - G_0\| < r$ implies $G \in U$. Let $c = G_0(p_0)$, and then use the hypothesis to select a special function g in $C[D]$ obeying (2). Let B be a closed ball in D, centered at p_0, such that

$$|G_0(p) - c| < \tfrac{r}{3} \quad \text{for all } p \in B, \tag{3}$$

and choose a smaller ball B_0, also centered at p_0, such that

$$|g(p) - c| < \tfrac{r}{3}, \qquad p \in B_0. \tag{4}$$

Construct a continuous function G_1 defined on B such that $G_1(p) = G_0(p)$ for p on the boundary of B, $G_1(p) = g(p)$ on B_0, and obeying $|G_1(p) - c| < \frac{r}{3}$ on B. Then, extend G_1 to all of D by setting it equal to G_0 off B. Observe that G_1 is in $C[D]$ and agrees with G_0 except on a small neighborhood of p_0 where it has been modified to agree with the special function g locally. If $p \in B$, we have

$$\begin{aligned} |G_1(p) - G_0(p)| &\le |G_1(p) - c| + |c - G_0(p)| \\ &\le \tfrac{r}{3} + \tfrac{r}{3} < r. \end{aligned} \tag{5}$$

Accordingly, $\|G_1 - G_0\|_D < r$ and $G_1 \in U$.

By (2), choose $\delta > 0$ so that $\|f - g\|_{B_0} > \delta$ for all $f \in \mathcal{F}$, and take $U_1 = \{\text{all } F \in U \text{ with } \|F - G_1\|_D < \frac{\delta}{2}\}$. Then, if $F \in U_1$ and $f \in \mathcal{F}$,

$$\begin{aligned} \|f - F\|_D &\ge \|f - g\|_{B_0} - \|F - G_1\|_{B_0} \\ &\ge \delta - \tfrac{\delta}{2} = \tfrac{\delta}{2}. \end{aligned}$$

Thus, $\mathcal{F}$ is nowhere dense in $C[D]$.

We now use this to show that $\mathcal{N}^2$ is nowhere dense in $C[I^2]$. Take $p_0 = (0,0)$; since $\mathcal{N}^2$ is closed under addition of constants, we do not need to retain the condition in (2) that $g(p_0) = c$. For any $r > 0$, let V_r be the compact neighborhood of p_0 consisting of those (x, y) with $|x| \le r$, $|y| \le r$.

THEOREM 2. *The special function $g(x,y) = x^2 + xy + y^2 + 2x + y$ has the property that*

$$r^2 > \inf_{f \in \mathcal{N}^2} \|f - g\|_{V_r} > r^3/10 \tag{6}$$

for all $r < .01$.

We begin the proof of this by quoting one of the characterization theorems for nomographic functions obtained in [3], modifying it to match the notation and needs of the present paper. (See Theorem 12, p. 293.)

Let g be of class C' on the set V_r, and suppose that g_x and g_y are bounded below by $\sigma > 0$. Let $\epsilon < r\sigma/12$ and suppose that the distance in the space $C[V_r]$ between g and the set $\mathcal{N}^2$ is less than ϵ. Then, one of the following systems of inequalities must be solvable.

(i) *For some choice of x_i in $[-r, r]$,*

$$\begin{aligned} |g(x_1, -r) - g(-r, 0)| &< 2\epsilon, \\ |g(x_2, -r) - g(-r, r)| &< 2\epsilon, \\ |g(x_1, r) - g(x_2, 0)| &< 2\epsilon. \end{aligned} \tag{7}$$

(ii) *For some choice of y_i in $[-r, r]$,*

$$\begin{aligned} |g(-r, y_1) - g(0, -r)| &< 2\epsilon, \\ |g(-r, y_2) - g(r, -r)| &< 2\epsilon, \\ |g(r, y_1) - g(0, y_2)| &< 2\epsilon. \end{aligned} \tag{8}$$

If we apply this general result to the special function $g(x, y)$ in Theorem 2, and assume that the distance from g to $\mathcal{N}^2$ is less than ϵ, then after putting $x_i = rs_i$ and $y_i = rt_i$, we have *either there are s_i with $|s_i| \leq 1$ and*

$$\begin{aligned} |2s_1 + 1 + (s_1^2 - s_1)r| &< 2\epsilon/r, \\ |2s_2 + (s_2^2 - s_2)r| &< 2\epsilon/r, \\ |2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r| &< 2\epsilon/r; \end{aligned} \tag{9}$$

or there exist t_i with $|t_i| \leq 1$ such that

$$\begin{aligned} |t_1 - 1 + (t_1^2 - t_1)r| &< 2\epsilon/r, \\ |t_2 - 3 + (t_2^2 - t_2)r| &< 2\epsilon/r, \\ |t_1 - t_2 + 2 + (t_1^2 - t_2^2 + t_1 + 1)r| &< 2\epsilon/r. \end{aligned} \tag{10}$$

We now show that if $\epsilon = r^3/10$ and $r < .01$, then neither (9) nor (10) can be satisfied. For (10) this is immediate, since the second inequality in (10) implies that

$$|t_2 - 3| \leq 2r + r^2/5,$$

contradicting $|t_1| \leq 1$. To show that (9) also fails, set

$$A = 2s_1 + 1 + (s_1^2 - s_1)r, \qquad B = 2s_2 + (s_2^2 - s_2)r,$$
$$C = 2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r,$$

so that the inequalities in (10) become $|A| < r^2/5$, $|B| < r^2/5$, $|C| < r^2/5$. The first of these implies $|2s_1 + 1| \leq 2r + r^2/5 < 3r$, and hence that $s_1 = -\frac{1}{2} + ar$ where $|a| \leq \frac{3}{2}$. Substituting this into A, we have $|2a + \frac{3}{4}| \leq 3r + \frac{r}{5} + \frac{9}{4}r^2 < 4r$ and conclude that $s_1 = -\frac{1}{2} - \frac{3}{8}r + br^2$ where $|b| \leq 2$. In a similar way, $|B| < r^2/5$ implies that $s_2 = cr^2$ where $|c| \leq 1$. Finally, since $C = A - B + (2s_1 + 1 - s_2)r$, we find $|2s_1 + 1 - s_2| \leq \frac{3}{5}r$ and using the values obtained for s_1 and s_2, we finally obtain

$$\left|-\tfrac{3}{4} + (2b - c)r\right| \leq \tfrac{3}{5}$$

which clearly cannot hold if $r < .01$.

To obtain the left side of (6), we merely observe that the special function $f_0(x, y) = \left(x + \frac{1}{2}y + 1\right)^2 - 1$ belongs to $\mathcal{N}^2$ and obeys $\|f_0 - g\|_{V_r} < r^2$.

Having proved Theorem 2, we invoke Theorem 1 and obtain

COROLLARY. *$\mathcal{N}^2$ is a nowhere dense subset of $C[I^2]$.*

To show now that this is also true of the class $\mathcal{N}^k$ in the space $C[I^k]$, we prove that one does not obtain a better nomographic approximation to the special function g given in Theorem 2 by using nomographic functions of k variables for any $k > 2$.

THEOREM 3. *If g is any continuous function of (x_1, x_2) then*

$$\inf_{f \in \mathcal{N}^k} \|f - g\|_{I^k} = \inf_{f^* \in \mathcal{N}^2} \|f^* - g\|_{I^2}. \tag{11}$$

PROOF. Let d be the distance in $C[I^k]$ from g to $\mathcal{N}^k$. Given $\delta > 0$, choose $f_0 \in \mathcal{N}^k$ so that

$$d_0 = \|f_0 - g\|_{I^k} < d + \delta.$$

Choose $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$ in I^k so that

$$d_0 = |f_0(\bar{x}) - g(\bar{x})| = |h(\phi_1(\bar{x}_1) + \phi_2(\bar{x}_2) + \cdots + \phi_k(\bar{x}_k)) - g(\bar{x})|.$$

Define a function ψ on I by

$$\psi(s) = \phi_2(s) + \phi_3(\bar{x}_3) + \cdots + \phi_k(\bar{x}_k),$$

and then set $f^*(x_1, x_2) = h(\phi_1(x_1) + \psi(x_2))$. It is then clear that

$$|f^*(\bar{x}_1, \bar{x}_2) - g(\bar{x}_1, \bar{x}_2)| = d_0$$

while for any (x_1, x_2) in I^2,

$$|f^*(x_1, x_2) - g(x_1, x_2)| = |f_0(x_1, x_2, \bar{x}_3, \ldots, \bar{x}_k) - g(x_1, x_2)| \leq \|f_0 - g\|_{I^k}.$$

Accordingly, $\|f^* - g\|_{I^2} = \|f_0 - g\|_{I^k} < d + \delta$, holding for every $\delta > 0$. This proves one half of (11); the rest follows since $\mathcal{N}^2$ is part of $\mathcal{N}^k$.

While five copies of $\mathcal{N}^2$ are enough to give $C[I^2]$ as their algebraic sum, it is known that four will not suffice. (See [5].) It would be of interest to know if the sum of four copies of $\mathcal{N}^2$ is perhaps also nowhere dense in $C[I^2]$. The argument we have used here will not work since the special function $g(x, y)$ used in Theorem 2 in fact is already a member of $\mathcal{N}^2 + \mathcal{N}^2$. Indeed,

$$g(x, y) = \exp(\log(x + 2) + \log(y + 3)) + (x^2 - x + y^2 - y - 6).$$

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