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HILBERTIAN INTERPRETATIONS OF MANUALS

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ABSTRACT. We characterize manuals which admit an interpretation in a manual on a Hilbert space. This characterization is given in terms of a certain set of states that the manual supports.

1. Introduction. In a series of papers [**1–4**, **8–10**], D. Foulis and C. Randall developed a mathematical formalism for quantum mechanics and other empirical sciences. Their formalism is at a more primitive level than the quantum logic approach [**5**, **6**, **7**, **11**] and, in fact, the latter can be derived from the former. In their approach, the physical operations form the basis of an axiomatic system in which the operations band together to form a mathematical structure called a manual.

A basic problem in axiomatic quantum mechanics is to characterize general quantum systems which are isomorphic to the traditional Hilbert space quantum mechanics. In this paper we give such a result for manuals. In particular, we show that a manual $\mathcal{A}$ admits a Hilbert space morphism if and only if $\mathcal{A}$ supports a collection of states of a certain type.

2. Definitions and notation. Most of the definitions in this section may be found in the work of Foulis and Randall cited in the introduction. Let X be a nonempty set and let $\mathcal{A} = \mathcal{A}(X)$ be a collection of nonempty subsets of X such that $X = \bigcup \mathcal{A}$. The elements of X are called *outcomes*, and the sets in $\mathcal{A}$ are called *operations*. Any subset of an operation is an *event*. Denote the set of events by $\mathcal{E}(\mathcal{A})$. If $x, y \in E \in \mathcal{A}$ and $x \neq y$ we write $x \perp y$. If $A \subseteq X$ we write

$$A^{\perp} = \{x \in X: x \perp y \; \forall y \in A\}$$

and for $A, B \subseteq X$ we write $A \perp B$ if $A \subseteq B^{\perp}$. We call $\mathcal{A}$ a *manual* if

(1) $E, F \in \mathcal{A}$ and $E \subseteq F$ implies $E = F$;

(2) $A, B \in \mathcal{E}(\mathcal{A})$ and $A \perp B$ implies $A \cup B \in \mathcal{E}(\mathcal{A})$.

A *morphism* ϕ from a manual $\mathcal{A}$ to a manual $\mathcal{B}$ is a map ϕ: $\mathcal{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{B})$ such that

(1) if $A_i \in \mathcal{E}(\mathcal{A})$ and $A = \bigcup A_i \in \mathcal{E}(\mathcal{A})$, then $\phi(A) = \bigcup \phi(A_i)$;

(2) if $A, B \in \mathcal{E}(\mathcal{A})$ and $A^{\perp} \subseteq B^{\perp}$, then $\phi(A)^{\perp} \subseteq \phi(B)^{\perp}$.

Let ϕ be a morphism from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$. Then ϕ is said to be

(a) *outcome preserving* if $x \in X$ implies $\phi(x) \in Y$;

(b) *operation preserving* if $E \in \mathcal{A}$ implies $\phi(E) \in \mathcal{B}$;

(c) *faithful* if $A, B \in \mathcal{E}(\mathcal{A})$ and $\phi(A) \perp \phi(B)$ implies $A \perp B$;

(d) *conditioning* if $A, B \in \mathcal{E}(\mathcal{A})$ and $A \perp B$ implies $\phi(A) \perp \phi(B)$;

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(e) *injective* if ϕ is outcome preserving and $\phi(x) = \phi(y)$ implies $x = y$;

(f) a *homomorphism* if $\{\phi(E): E \in \mathcal{A}\}$ is a manual;

(g) a *homomorphism onto* if $\mathcal{B} = \{\phi(E): E \in \mathcal{A}\}$;

(h) an *isomorphism* if ϕ is injective and a homomorphism onto $\mathcal{B}$.

An operation preserving, conditioning morphism is an *interpretation.*

A *state* for a manual $\mathcal{A}$ is a map $\alpha: \mathcal{E}(\mathcal{A}) \to [0,1]$ such that

(1) if $E \in \mathcal{A}$, then $\alpha(E) = 1$;

(2) if $A, B \in \mathcal{E}(\mathcal{A})$ and $A \perp B$, then $\alpha(A \cup B) = \alpha(A) + \alpha(B)$.

A state α for $\mathcal{A}$ is *regular* if for any family $A_i \in \mathcal{E}(\mathcal{A})$ with $A_i \perp A_j$, $i \neq j$, such that $A = \bigcup A_i \in \mathcal{E}(\mathcal{A})$ we have $\alpha(A) = \sum \alpha(A_i)$. We denote the set of regular states on $\mathcal{A}$ by $\sum(\mathcal{A})$. If $\alpha \in \sum(\mathcal{A})$, $x \in X$, we write $\alpha(x) = \alpha(\{x\})$. It is clear that the function $x \to \alpha(x)$ determines α. A function $f: X \to \mathbf{C}$ is an *amplitude* function [**12**] for $\alpha \in \sum(\mathcal{A})$ if $|f(x)|^2 = \alpha(x)$ for all $x \in X$. Clearly, any $\alpha \in \sum(\mathcal{A})$ has many amplitude functions. Indeed, $f(x) = \lambda\alpha(x)^{1/2}$ is such a function for any $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. We call a function $f: X \to \mathbf{C}$ *summable* if $\sum_{x \in E} f(x)$ exists for all $E \in \mathcal{A}$ and $\sum_{x \in E} f(x) = \sum_{x \in F} f(x)$ for all $E, F \in \mathcal{A}$. An example of a summable function is $x \to \alpha(x)$ for any $\alpha \in \sum(\mathcal{A})$.

We now give an example of a manual which is of importance to our present work. Let H be a complex Hilbert space and let $S(H) = \{v \in H: \|v\| = 1\}$ be its unit sphere. Let H_1 be the set of all one-dimensional (orthogonal) projections on H and let $\mathcal{A}(H_1)$ be the collection of all maximal orthogonal sets in H_1. Then $\mathcal{A}(H_1)$ is a manual. We call $\mathcal{A}(H_1)$ the *Hilbertian* manual on H. Our Hilbertian manual is a submanual of the Foulis-Randall *Hilbert space manual* [**4**] consisting of the collection of all maximal orthogonal sets of projections on H. Both these manuals generate the same "operational logic" [**4**]. It is frequently convenient to consider the elements of H_1 as one-dimensional subspaces of H.

What distinguishes a Hilbertian manual $\mathcal{A}(H_1)$ from among others in the general class of manuals? The Hilbertian manuals support a special set of states. Corresponding to a $v \in S(H)$ we define the *vector state* $\hat{v}$ by $\hat{v}(p) = \langle pv, v\rangle$ for all $p \in H_1$. Let $\mathcal{V}(H) = \{\hat{v}: v \in S(H)\}$. The set of vector states $\mathcal{V}(H)$ has two important properties. First, if $\emptyset \neq A \in \mathcal{E}(\mathcal{A})$, then there exists a $\hat{v} \in \mathcal{V}(H)$ such that $\hat{v}(A) = 1$. For the second property, for each $p \in H_1$ choose a $\tilde{p} \in S(H)$ such that $\tilde{p} \in p$. If $E \in \mathcal{A}(H_1)$, it is clear that $\{\tilde{p}: p \in E\}$ is an orthonormal basis for H. Now for each $\hat{v} \in \mathcal{V}(H)$ define the amplitude function $f_v: H_1 \to \mathbf{C}$ by $f_v(p) = \langle v, \tilde{p}\rangle$. Then for any $\hat{u}, \hat{v} \in \mathcal{V}(H)$, the function $f_u\overline{f}_v$ is summable. Indeed, if $E, F \in \mathcal{A}(H_1)$ we have

$$\sum_{p \in E} f_u(p)\overline{f}_v(p) = \sum_{p \in E} \langle u, \tilde{p}\rangle\langle \tilde{p}, v\rangle = \langle u, v\rangle = \sum_{p \in F} \langle u, \tilde{p}\rangle\langle \tilde{p}, v\rangle = \sum_{p \in F} f_u(p)\overline{f}_v(p).$$

We close this section by defining families of states with certain special properties. Let $\mathcal{A}(X)$ be a manual and let $\Delta \subseteq \sum(\mathcal{A})$. Then Δ is said to be

(1) *unital* if for any $\emptyset \neq A \in \mathcal{E}(\mathcal{A})$ there exists an $\alpha \in \Delta$ such that $\alpha(A) = 1$;

(2) *strongly separating* if $x, y \in X$ and $x \neq y$ implies that there exists an $\alpha \in \Delta$ such that $\alpha(x) \neq \alpha(y)$;

(3) *strongly $\perp$-determining* if $\{\alpha \in \Delta: \alpha(A) = 1\} \subseteq \{\alpha \in \Delta: \alpha(B) = 0\}$ implies $A \perp B$, $A, B \in \mathcal{E}(\mathcal{A})$;

(4) a set of *amplitude states* if for every $\alpha \in \Delta$ there is an amplitude function f_α for α such that for every $\alpha, \beta \in \Delta$, $f_\alpha\overline{f}_\beta$ is summable.

We have shown above that $\mathcal{V}(H)$ is a unital set of amplitude states. It is also straightforward to show that $\mathcal{V}(H)$ is strongly separating and strongly $\perp$-determining.

3. Hilbertian interpretations. We now characterize those manuals which have a Hilbertian interpretation.

THEOREM. *A manual $\mathcal{A}(X)$ has an outcome preserving interpretation ϕ in a Hilbertian manual $\mathcal{A}(H_1)$ if and only if $\mathcal{A}(X)$ admits a unital set of amplitude states Δ. Moreover, if the above condition holds, then there exists an injection $\psi\colon \Delta \to \mathcal{V}(H)$ such that $\alpha(A) = \psi(\alpha)[\phi(A)]$ for all $A \in \mathcal{E}(\mathcal{A})$.*

PROOF. Suppose $\phi\colon \mathcal{E}(\mathcal{A}) \to \mathcal{E}[\mathcal{A}(H_1)]$ is an outcome preserving interpretation. Then for every $A \in \mathcal{E}(\mathcal{A})$, $\phi(A) = \bigcup_{x\in A} \phi(x)$. For each $\hat{v} \in \mathcal{V}(H)$, $A \in \mathcal{E}(\mathcal{A})$, define $\alpha_v(A) = \hat{v}[\phi(A)] = \sum_{x\in A} \hat{v}[\phi(x)]$. It is straightforward to show that each α_v is a regular state on $\mathcal{A}(X)$. Let $\Delta = \{\alpha_v\colon \hat{v} \in \mathcal{V}(H)\}$. To show that Δ is unital, let $\emptyset \neq A \in \mathcal{E}(\mathcal{A})$ and let $p \in \phi(A)$. Then there exists a $v \in S(H)$ such that $v \in p$. Hence, $\alpha_v(A) = \hat{v}[\phi(A)] = 1$. To show that Δ is a set of amplitude states, for each $p \in H_1$ choose a $\tilde{p} \in S(H)$ such that $\tilde{p} \in p$. For $\alpha_v \in \Delta$, define $f_v\colon X \to \mathbf{C}$ by $f_v(x) = \langle v, \phi(x)^{\sim}\rangle$. Then f_v is an amplitude function for α_v since

$$|f_v(x)|^2 = |\langle v, \phi(x)^{\sim}\rangle|^2 = \hat{v}[\phi(x)] = \alpha_v(x).$$

Finally, for any $E, F \in \mathcal{A}(X)$, $\alpha_u, \alpha_v \in \Delta$ we have

$$\begin{aligned}\sum_{x\in E} f_u(x)\bar{f}_v(x) &= \sum_{x\in E}\langle u, \phi(x)^{\sim}\rangle\langle \phi(x)^{\sim}, v\rangle = \langle u, v\rangle \\ &= \sum_{x\in F}\langle u, \phi(x)^{\sim}\rangle\langle \phi(x)^{\sim}, v\rangle = \sum_{x\in F} f_u(x)\bar{f}_v(x).\end{aligned}$$

Conversely, let Δ be a unital set of amplitude states on $\mathcal{A}(X)$ and let $H_0 = \{f_\alpha\colon \alpha \in \Delta\}$ be the corresponding set of amplitude functions. Let $[H_0]$ be the linear span of H_0. For $f, g \in [H_0]$ define $\langle f, g\rangle = \sum_{x\in E} f(x)\bar{g}(x)$ where $E \in \mathcal{A}$. It is straightforward to show that $\langle f, g\rangle$ is well defined and independent of $E \in \mathcal{A}$. It is clear that $\langle\cdot,\cdot\rangle$ is an inner product. Let H be the Hilbert space completion of $[H_0]$ relative to the inner product $\langle\cdot,\cdot\rangle$. We now show that if $f \in H$, then $f\colon X \to \mathbf{C}$ and $\|f\|^2 = \sum_{x\in E} |f(x)|^2 = \sum_{x\in F} |f(x)|^2$ for every $E, F \in \mathcal{A}$. Moreover, if $f, g \in H$, then $\langle f, g\rangle = \sum_{x\in E} f(x)\bar{g}(x) = \sum_{x\in F} f(x)\bar{g}(x)$ for every $E, F \in \mathcal{A}$. Indeed, let f_i be a Cauchy sequence in $[H_0]$. Then $f_i(x)$ converges in $\mathbf{C}$ for every $x \in X$ and hence there exists an $f\colon X \to \mathbf{C}$ such that $f_i(x) \to f(x)$ for all $x \in X$. It follows that $\|f - f_i\| \to 0$ as $i \to \infty$. Also,

$$\begin{aligned}\|f\|^2 &= \lim \|f_i\|^2 = \lim \sum_{x\in E} |f_i(x)|^2 = \lim \sum_{x\in F} |f_i(x)|^2 \\ &= \sum_{x\in E} |f(x)|^2 = \sum_{x\in F} |f(x)|^2.\end{aligned}$$

The second statement now follows from the polarization identity.

For $x \in X$, let $\phi(x)$ be the linear span $[\{f_\alpha\colon \alpha(x) = 1, \alpha \in \Delta\}] \subseteq H$. We now show that $\phi(x)$ is one-dimensional and hence in H_1. Suppose $\alpha(x) = \beta(x) = 1$ for $\alpha, \beta \in \Delta$. If $y \perp x$, then $\alpha(y) = \beta(y) = 0$ and hence, $f_\alpha(y) = f_\beta(y) = 0$. Thus, if

$x \in E \in \mathcal{A}$ we have

$$|\langle f_\alpha, f_\beta\rangle| = \left|\sum_{y\in E} f_\alpha(y)\overline{f}_\beta(y)\right| = |f_\alpha(x)|\,|f_\beta(x)| = 1 = \|f_\alpha\|\,\|f_\beta\|.$$

Since we have an equality in Schwarz's inequality, there is a $\lambda \in \mathbf{C}$ such that $f_\beta = \lambda f_\alpha$. Hence, ϕ: $X \to H_1$ and ϕ preserves outcomes. Extend ϕ to $\mathcal{E}(\mathcal{A})$ by defining $\phi(A) = \bigcup_{x\in A}\phi(x)$ for $A \in \mathcal{E}(\mathcal{A})$. If $E \in \mathcal{A}$, we now show that $\phi(E) \in \mathcal{A}(H_1)$. Let $x, y \in E$ with $x \neq y$ and suppose $\alpha(x) = \beta(y) = 1$, $\alpha, \beta \in \Delta$. Then

$$\langle f_\alpha, f_\beta\rangle = \sum_{z\in E} f_\alpha(z)\overline{f}_\beta(z) = \sum_{z\in E}\delta_{xz}\delta_{yz} = 0.$$

Hence, $\phi(x)\perp\phi(y)$. Now let $p \in H_1$ satisfy $p\perp\phi(E)$. If $g \in p$, $x \in E$ and $\alpha(x) = 1$, then $g\perp f_\alpha$. Hence,

$$0 = \langle g, f_\alpha\rangle = \sum_{y\in E} g(y)\overline{f}_\alpha(y) = g(x)\overline{f}_\alpha(x).$$

Hence, $g(x) = 0$ for all $x \in E$ and $\|g\|^2 = \sum_{x\in E}|g(x)|^2 = 0$. Thus, $g = 0$. It follows that $\phi(E) \in \mathcal{A}(H_1)$ so ϕ preserves operations and events. Hence, ϕ: $\mathcal{E}(\mathcal{A}) \to \mathcal{E}[\mathcal{A}(H_1)]$. Moreover, if $x\perp y$ then $\phi(x)\perp\phi(y)$ so $A, B \in \mathcal{E}(\mathcal{A})$ and $A\perp B$ implies $\phi(A)\perp\phi(B)$.

We now show that ϕ is a morphism and therefore an interpretation (actually this follows from [4, Lemma 2], but we shall give the proof to make this work self-contained). First, condition (1) in the definition of a morphism clearly holds. For condition (2), suppose $A, B \in \mathcal{E}(\mathcal{A})$ and $A^\perp \subseteq B^\perp$. Let $p \in \phi(A)^\perp$ and $q \in \phi(B)$ and let $f, g \in H$ satisfy $f \in p$, $g \in q$. Let $E \in \mathcal{A}$ satisfy $A \subseteq E$. Then $E - A \subseteq A^\perp \subseteq B^\perp$ and hence

$$\phi(E - A) \subseteq \phi(B^\perp) \subseteq \phi(B)^\perp \subseteq q^\perp.$$

Hence $q \in \phi(E - A)^\perp$. Since $\sum_{x\in E}\phi(x) = I$, we have $f = [\sum_{x\in E-A}\phi(x)]f$ and $g = [\sum_{x\in A}\phi(x)]g$. Hence, $f\perp g$ so $p\perp q$ and $\phi(A)^\perp \subseteq \phi(B)^\perp$.

For the last statement of the theorem, define ψ: $\Delta \to \mathcal{V}(H)$ by $\psi(\alpha) = \hat{f}_\alpha$. Let $x \in X$ and let $\beta \in \Delta$ satisfy $\beta(x) = 1$. Then for any $\alpha \in \Delta$ we have

$$\psi(\alpha)[\phi(x)] = \hat{f}_\alpha[\phi(x)] = \langle\phi(x)f_\alpha, f_\alpha\rangle = |\langle f_\alpha, f_\beta\rangle|^2 = |f_\alpha(x)|^2 = \alpha(x).$$

Hence, for any $A \in \mathcal{E}(\mathcal{A})$ we have

$$\psi(\alpha)[\phi(A)] = \psi(\alpha)\left[\bigcup_{x\in A}\phi(x)\right] = \sum_{x\in A}\psi(\alpha)[\phi(x)] = \sum_{x\in A}\alpha(x) = \alpha(A). \quad \square$$

COROLLARY. *A manual $\mathcal{A}(X)$ has an injective interpretation in a Hilbertian manual if and only if $\mathcal{A}(X)$ admits a unital, strongly separating set of amplitude states.*

PROOF. Suppose Δ is a unital, strongly separating set of amplitude states on $\mathcal{A}(X)$. Let ϕ: $X \to H_1$ and ψ: $\Delta \to \mathcal{V}(H)$ be the maps constructed in the theorem. To show that ϕ is injective assume that $\phi(x) = \phi(y)$. Then for any $\alpha \in \Delta$ we have

$$\alpha(y) = \psi(\alpha)[\phi(y)] = \psi(\alpha)[\phi(x)] = \alpha(x).$$

Hence $x = y$. Conversely, suppose $\phi\colon \mathcal{E}(\mathcal{A}) \to \mathcal{E}[\mathcal{A}(H_1)]$ is an injective interpretation in a Hilbertian manual $\mathcal{A}(H_1)$. Define $\Delta = \{\alpha_v : \hat{v} \in \mathcal{V}(H)\}$ as in the theorem. The theorem shows that Δ is a unital set of amplitude states on $\mathcal{A}$. To show that Δ is strongly separating assume that $\alpha(x) = \alpha(y)$ for all $\alpha \in \Delta$. Then $\hat{v}[\phi(x)] = \hat{v}[\phi(y)]$ for all $\hat{v} \in \mathcal{V}(H)$. Since $\mathcal{V}(H)$ is strongly separating on $\mathcal{A}(H_1)$ we have $\phi(x) = \phi(y)$. Since ϕ is injective, $x = y$. $\square$

COROLLARY. *A manual $\mathcal{A}$ is isomorphic to a submanual of a Hilbertian manual if and only if $\mathcal{A}$ admits a unital, strongly separating set of amplitude states.*

COROLLARY. *A manual $\mathcal{A}$ has an outcome preserving faithful interpretation in a Hilbertian manual if and only if $\mathcal{A}$ admits a unital, strongly $\perp$-determining set of amplitude states.*

COROLLARY. *If a manual $\mathcal{A}$ admits a unital set of amplitude states Δ, then $\mathcal{A}$ has an outcome preserving interpretation in a Hilbert space manual.*

Corresponding results hold in the last corollary if Δ is separating or strongly $\perp$-determining.

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