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A NOTE ON INFINITE LOOP SPACE MULTIPLICATIONS

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ABSTRACT. A monoid M is known to be abelian iff its multiplication $M \times M \to M$ is a homomorphism. We prove the corresponding result for homotopy-everything H-spaces, e.g. infinite loop spaces: For a homotopy-everything H-space X each n-ary operation $X^n \to X$ is a homotopy homomorphism, i.e. a homomorphism up to homotopy and all higher coherence *conditions*.

In [**1**] and [**2**] J. M. Boardman and I proved that an H-space X is an infinite loop space iff its multiplication enjoys nice properties concerning associativity and commutativity. These properties were described in terms of universal algebra, and the necessary and sufficient condition for X to be an infinite loop space essentially boils down to the fact that the morphism spaces $\mathcal{E}(n,1)$ of the PROP $\mathcal{E}$ encoding the H-structure of X be contractible (for the definition of a PROP see [**2**, Definition 2.44]). Dropping all unnecessary structure of a PROP, P. May in [**4**] introduced the simpler notion of an operad and obtained the corresponding result on infinite loop space structures more directly. Using his terminology we call $\mathcal{E}$ and E_∞-PROP if each $\mathcal{E}(n,1)$ is contractible and Σ-free if the operation of the symmetric group Σ_n on $\mathcal{E}(n,1)$ makes $\mathcal{E}(n,1)$ a numerable principal Σ_n-space. An E_∞-space X is an H-space whose structure is given by an action of an E_∞-PROP on X.

Let $\mathcal{E}$ be an E_∞-PROP and X an $\mathcal{E}$-space. There is a canonical product action of $\mathcal{E}$ on the k-fold product X^k. Each element $\chi \in \mathcal{E}(k,1)$ defines a map $\chi: X^k \to X$. It is the purpose of this note to show

THEOREM. *Suppose $\mathcal{E}$ is a Σ-free E_∞-PROP or each $\mathcal{E}(n,1)$ is Σ_n-equivariantly contractible. Let $\chi \in \mathcal{E}(k,1)$ and X be an $\mathcal{E}$-space. Then $\chi: X^k \to X$ can be extended to a homotopy $\mathcal{E}$-map in the sense of* [**2**, *Definition* 4.2].

T. Lada tried in [**3**] to prove a result of this kind but only succeeded in the case $k = 2$ and $\mathcal{E} = \mathcal{Q}$, the little cubes PROP of [**2**,2.49]. His proof is given by a number of explicit formulas depending on the geometry of the spaces of little cubes. Our proof of the theorem is an easy consequence of the theory of [**1**] and [**2**]. We use the terminology of [**2**, Chapter III, IV].

Let $\mathcal{E}$ be an arbitrary PROP and X and $\mathcal{E}$-space. Then $\mathcal{E}$ operates on X^k by

$$\alpha : (X^k)^n \xrightarrow{\tau_{n,k}} (X^n)^k \xrightarrow{\alpha^k} X^k,$$

$\alpha \in \mathcal{E}(n,1)$, where $\tau_{n,k}$ is the homeomorphism

$$((x_{11},\ldots,x_{1k}),\ldots,(x_{n1},\ldots,x_{nk})) \to ((x_{11},\ldots,x_{n1}),\ldots,(x_{1k},\ldots,x_{nk})).$$

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To prove the theorem we need an action of $HW(\mathcal{E}\otimes\mathcal{L}_1)$ extending χ and the actions on X^k and X. The idea is to apply the Lifting Theorem [**2**, Theorem 3.7] with $\mathcal{V}\subset HW(\mathcal{E}\otimes\mathcal{L}_1)$ being the subcategory generated under composition and $\oplus$ by $W(\mathcal{E}\otimes\{0\})$, $W(\mathcal{E}\otimes\{1\})$ and the single morphism $(\mathrm{id}_1\otimes(0\to 1))$. To minimize notation we denote the objects $(n,0)$ and $(n,1)$ in $\mathcal{E}\times\mathcal{L}_1$ by n respectively n'. For $\mathcal{C}$ we take the PROP $\mathcal{S}\otimes\mathcal{L}_1$, where $\mathcal{S}$ is the PROP of abelian monoids, i.e. $\mathcal{S}(n,1)$ consists of a single point for all n. We have to construct an appropriate PROP $\mathcal{D}$ acting on the pair (X^k,X) extending the $\mathcal{E}$-action on X^k and on X and the map χ, yielding a commutative diagram:

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, hook'] \arrow[dr, hook] & \\ HW(\mathcal{E}\otimes\mathcal{L}_1) \arrow[d, "\epsilon"] & & \mathcal{D} \arrow[d, "G"] \\ \mathcal{E}\otimes\mathcal{L}_1 \arrow[rr, "F"] & & \mathcal{S}\otimes\mathcal{L}_1 \end{tikzcd}$$

Here F and G are the uniquely determined PROP-functors. If each $\mathcal{D}(n,1)$, $\mathcal{D}(n,1')$ and $\mathcal{D}(n',1')$ is contractible (respectively Σ_n-equivariantly contractible), and $\mathcal{E}$ is Σ-free (respectively arbitrary) the theorem is proved.

Construction of $\mathcal{D}$. It suffices to specify

(A) $\qquad \mathcal{D}(n,1):=\mathcal{E}(n,1),\quad \mathcal{D}(n,1'):=\mathcal{E}(nk,1),\quad \mathcal{D}(n',1'):=\mathcal{E}(n,1),$

the composite of a morphism α in these spaces with a permutation $\pi\in\Sigma_n$, and of α with an n-fold sum $\beta_1\oplus\ldots\oplus\beta_n$ with β_i in the appropriate spaces (A). As long as we stick in the full subcategories of objects n respectively n' these compositions are given by the composition in $\mathcal{E}$. If $\alpha\in\mathcal{D}(n',1')$ and $\beta_i\in\mathcal{D}(n_i,1')$ the composition is again the one in $\mathcal{E}$. It remains to define $\alpha\circ\pi$ and $\alpha\circ(\beta_1\otimes\cdots\otimes\beta_n)$ for $\alpha\in\mathcal{D}(n,1')=\mathcal{E}(nk,1)$, $\beta_i\in\mathcal{D}(n_i,1)=\mathcal{E}(n_i,1)$:

$$\alpha\circ\pi=\alpha\circ(\pi\oplus\cdots\oplus\pi),$$
$$\alpha\circ(\beta_1\oplus\cdots\oplus\beta_n)=\alpha\circ[(\beta_1\oplus\cdots\oplus\beta_n)\oplus\cdots\oplus(\beta_1\oplus\cdots\oplus\beta_n)],$$

where on the right side we have composition in $\mathcal{E}$ with k summands π respectively $(\beta_1\oplus\cdots\oplus\beta_n)$.

The operation of $\mathcal{D}$ on the pair (X^k,X) is given as follows.

$\alpha\in\mathcal{D}(n,1)=\mathcal{E}(n,1)$ operates as $\alpha\circ\tau_{n,k}\colon(X^k)^n\to X^k$,

$\alpha\in\mathcal{D}(n,1')=\mathcal{E}(nk,1)$ operates as $\alpha\circ\tau_{n,k}\colon(X^k)^n\to X$,

$\alpha\in\mathcal{D}(n',1')=\mathcal{E}(n,1)$ operates as $\alpha\colon X^n\to X$.

Obviously, $\mathcal{D}$ satisfies all the requirements listed above, which proves the theorem.

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