PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 1, September 1982

ON GRADED RINGS WITH FINITENESS CONDITIONS

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ABSTRACT. It is proved that a graded ring that is finitely graded modulo its radical is local if its initial subring is local, and that a graded artinian ring is finitely generated over its initial subring which is also artinian. These results extend theorems of Gordon and Green on artin algebras. Other results relating the structure of a graded ring to that of its initial subring are also presented.

In this note we provide simple proofs of extensions of the principal results in R. Gordon and E. Green's recent paper [**4**] on graded artin algebras. First we prove that any finitely graded (modulo the radical) ring with local initial subring is itself local. From this it follows that over any graded ring a finitely graded module with a composition series is an indecomposable module if and only if it is indecomposable as a graded module. Then we prove that a graded ring is left artinian if and only if it has a composition series as a left module over its initial subring. These results are used to show that over a graded left artinian ring every simple module, every projective module, and every injective left module is isomorphic to a graded module, as is every direct summand of a finitely generated graded left module. Also they yield information about the relative structure of a graded ring and its initial subring, for example, a finitely graded ring is semiprimary if and only if so is its initial subring.

Recall (see [**3**, **6**]) that a ($\mathbf{Z}$-) *graded ring* is a ring R together with an abelian group decomposition $R = \oplus_{\mathbf{Z}} R_n$ such that $R_n R_m \subseteq R_{n+m}$ $(n, m \in \mathbf{Z})$, and that R_0 is a unital subring called the *initial subring* of R. We let $J = J(R)$ denote the radical of R and $J_0 = J(R_0)$. Our first result, generalizing [**4**, Theorem 3.1], requires no more than this.

1. THEOREM. *Let $R = \oplus_{\mathbf{Z}} R_n$ be a graded ring. If R_0 is local and $R_n \subseteq J$ for all but finitely many $n \in \mathbf{Z}$, then R is a local ring.*

PROOF. By hypothesis there is an $N \geq 0$ such that $R_n \subseteq J$ whenever $|n| > N$. Thus each R_n with $n \neq 0$ consists of zero divisors modulo J, and it follows that $R_n R_{-n}$ contains no units. Therefore, since R_0 is local, $R_n R_{-n} \subseteq J_0$ whenever $n \neq 0$. Let $k > 0$ and suppose, inductively, that $R_n \subseteq J$ whenever $|n| > k$. Let $b_k \in R_k$, $a = \Sigma a_n \in R$, and $x = 1 - b_k a$. To see that $b_k \in J$ we show that x is invertible. Since

Received by the editors September 8, 1981.
1980 *Mathematics Subject Classification*. Primary 16A03, 16A35; Secondary 16A10, 16A51.

$|n-k|>k$ when $n<0$,

$$\begin{aligned} x &= 1 - \sum_{n<0} b_k a_{n-k} - b_k a_{-k} - \sum_{n>0} b_k a_{n-k} \\ &= 1 - j - j_0 - p \end{aligned}$$

with $j \in J$, $j_0 \in J_0$ and $p \in \Sigma_{n>0} R_n$. Let $u_0 = (1-j_0)^{-1} \in R_0$. Then

$$xu_0 = 1 - j' - p'$$

with $j' \in J$ and $p' \in \Sigma_{n>0} R_n$. But then p' is nilpotent modulo J and so $(1-p')^{-1} = v$ exists in R. Now we have

$$xu_0v = 1 - j''$$

with $j'' \in J$ so xu_0v, and hence x, is (right) invertible. Thus $R_k \subseteq J$, and similarly $R_{-k} \subseteq J$, and we have shown that $R_n \subseteq J$ whenever $n \neq 0$. It follows that $r_0 \mapsto r_0 + J$ defines a surjective ring homomorphism $R_0 \to R/J$ so R is local.

A left R-module M is *graded* [**3**, **6**] in case it has a decomposition $M = \oplus_{\mathbf{Z}} M_n$ over R_0 such that $R_m M_n \subseteq M_{m+n}$ $(m, n \in \mathbf{Z})$. If $N = \oplus_{\mathbf{Z}} N_n$ is also a graded module, the elements of $\operatorname{Hom}_R(M,N)_n = \{f \in \operatorname{Hom}_R(M,N) \mid f(M_k) \subseteq N_{k+n}\ (k \in \mathbf{Z})\}$ are called *degree n homomorphisms*. The category R-Gr consisting of graded left R-modules and degree zero homomorphisms is a Grothendieck category [**5**]. The images (kernels) of degree zero homomorphisms to (from) M are the *homogeneous submodules* of M. If all but finitely many terms in the grading of a graded module or ring are zero then it is said to be *finitely graded*. Now Theorem 1 allows us to free [**4**, Theorem 3.2] of any finiteness conditions on its graded ground ring.

2. COROLLARY. *Let M be a finitely graded R-module with a.c.c. and d.c.c. on homogeneous submodules. Then M is indecomposable in R-*Gr *if and only if M is indecomposable in R-*Mod.

PROOF. One implication is obvious. For the other let $S = \operatorname{End}({}_R M)$. Then S is finitely graded with initial subring S_0 (see [**5**, Lemma 3.3.2, p. 83]), the ring of zero degree endomorphisms of M. If $s_0 \in S_0$ then by Fitting's Lemma $M = \operatorname{Im} s_0^c \oplus \operatorname{Ker} s_0^c$ where c is the composition length of M. But these are homogeneous submodules of M, so s_0 is invertible or nilpotent. Thus S_0 is local and so, by Theorem 1, is S.

According to [**6**, Lemma 3.3.6, p. 9] if any two maps in an equation $f = gh$ of R-maps between graded R-modules are of degree zero, then the third can be replaced by a degree zero homomorphism. It follows that any homogeneous R-direct summand of a graded R-module has at least one homogeneous complement, and that graded projective (injective) (semisimple) R-modules are projective (injective) (semisimple) in R-Gr.

Let $M = \oplus_{\mathbf{Z}} M_n$ be a graded left R-module and let $k \in \mathbf{Z}$. For each R_0-submodule $N_k \leq_{R_0} M_k$ let $\theta_k(N_k) = RN_k$, and for each homogeneous submodule $N = \oplus_{\mathbf{Z}} N_n$ of M let $\varphi_k(N) = N_k$. Then θ_k and φ_k define order preserving maps between the lattices of R_0 submodules of M_k and homogeneous submodules of M.

3. LEMMA. *The order preserving maps θ_k and φ_k satisfy*

$$\varphi_k\theta_k(N_k) = N_k \qquad (N_k \leqslant_{R_0} M_k).$$

Moreover $\varphi_k(N)$ is a direct summand of M_k whenever N is a homogeneous direct summand of M.

PROOF. If $N_k \leqslant_{R_0} M_k$ then $\theta_k(N_k) = RN_k = \bigoplus_{n\in\mathbf{Z}} R_{n-k}N_k$ and $\varphi_k\theta_k(N_k) = R_0N_k = N_k$. If $N = \bigoplus_{\mathbf{Z}} N_n$ is a homogeneous direct summand of M then N has a homogeneous complement $N' = \bigoplus N'_n$. But then clearly $M_k = \varphi_k(N \oplus N') = N_k \oplus N'_k$.

When discussing kinds of graded rings, our other descriptive adjectives apply directly to the ring in question, irregardless of its grading. Thus a graded semisimple ring is a semisimple ring that happens to be graded, and graded division rings are trivially graded, i.e., $R = R_0$ (see [**6**, Lemma 6.1, p. 58]). The following results show that, up to automorphism, semisimple rings can only be graded in finitely many ways.

4. PROPOSITION. *If $R = \bigoplus_{\mathbf{Z}} R_n$ is a graded semisimple ring then R_0 is semisimple and contains a complete orthogonal set $e_1,\dots,e_m$ of primitive idempotents of R such that for all $i, j \in \{1,\dots,m\}$ there is an $n \in \mathbf{Z}$ with $e_iRe_j \subseteq R_n$.*

PROOF. First assume that R_0 is a division ring. Since semisimple rings are von Neumann regular, if $0 \neq v_n \in R_n$ then there is an $a = \sum a_i \in R$ such that

$$r_n = r_n a r_n = \sum_i r_n a_{i-2n} r_n,$$

so we have $r_n = r_n a_{-n} r_n$. But then, since $a_{-n}r_n \in R_0$, a division ring, and $r_n \neq 0$ we see that $a_{-n}r_n = 1$. Thus every nonzero homogeneous element of R is invertible, and this implies that R is a (noncommutative) domain. Thus R is a division ring and $R = R_0$.

Now suppose that R is semisimple. Then so, by Lemma 3, is R_0. Let $e_1,\dots,e_m$ be a complete set of orthogonal primitive idempotents of R_0. Then $e_iR_0e_i$, the initial subring of the graded semisimple ring $e_iRe_i = \bigoplus_n e_iR_ne_i$, is a division ring, so

$$e_iRe_i = e_iR_0e_i \subseteq R_0$$

for $i = 1,\dots,m$. Next let $e_iae_j \in e_iRe_j$ and write

$$e_iae_j = \sum_k a_k = \sum_k e_ia_ke_j.$$

If some $e_ia_ne_j \neq 0$ then, since e_iRe_j is one-dimensional over $e_iRe_i = e_iR_0e_i$, for each $k \in \mathbf{Z}$ there is an $e_ib_0e_i \in e_iR_0e_i$ such that

$$e_ia_ke_j = e_ib_0e_ia_ne_j \in R_n.$$

But then $e_iRe_j \subseteq R_n$, and the proof is complete.

Now employing G. Bergman's recent theorem [**2**]: *The Jacobson radical of a graded ring R is a homogeneous ideal of R* (which incidentally proves the converse of

Theorem 1), we can easily prove

5. THEOREM. *A graded ring $R = \bigoplus_{\mathbf{Z}} R_n$ is left artinian if and only if R_0 is left artinian and R is finitely generated as a left R_0-module.*

PROOF. Let R be left artinian. By Proposition 4, $R/J = \bigoplus_{\mathbf{Z}} R_n + J/J$ is finitely graded, so by Lemma 3 R/J has a composition series over $R_0 + J/J$. Thus every simple R-module, and hence R, has a composition series over R_0. The converse is obvious.

This theorem generalizes [**4**, Lemma 1.3] from the commutative case and provides an effective analogue to [**4**, Theorem 1.4] by showing that graded finitely generated modules over a graded artinian ring are finitely graded. Thus by Theorems 2 and 5 we have

6. COROLLARY. *Let R be a graded left artinian ring. A graded finitely generated R-module is indecomposable in R-*Gr *if and only if it is indecomposable in R-*Mod.

Gordon and Green call an R-module *gradable* in case it is R-isomorphic to a module in R-Gr. The list of gradable modules over a graded artin algebra, given in [**4**, §3], is also valid in the left artinian case.

7. PROPOSITION. *If R is a graded left artinian ring then every left R-module that is semisimple, projective or injective is gradable, and so are all direct summands of finitely generated gradable modules.*

PROOF. The last statement follows from Corollary 6 and the Krull-Schmidt Theorem; and it implies that finitely generated, hence all, semisimple and projective left R-modules are gradable.

To see that injective modules are gradable, suppose R is graded $R = \bigoplus_{-N}^{N} R_n$, let C_0 be an injective cogenerator in R_0-Mod, and let

$$C = \mathrm{Hom}_{R_0}(R_R, C_0) = \bigoplus_{-N}^{N} \mathrm{Hom}_{R_0}(R_{-n}, C_0)$$

to obtain a finitely graded injective cogenerator C [**1**, Exercises 19.14, 19.20] in R-Mod. Now we note that $\mathrm{Soc}_R C = \mathrm{ann}_C(J)$ is a homogeneous submodule of C that is semisimple and essential in C as an object of R-Gr, that simple objects in R-Gr are simple in R-Mod (i.e., isomorphic to one of the Re_i of Proposition 4), and we observe that a minor modification of the usual proof in R-Mod shows that, since R has a.c.c. on homogeneous left ideals, direct sums of injectives are injective in R-Gr. Using these facts we have $C = \amalg E(S_\alpha)$ in R-Gr, where S_α are R-simple homogeneous submodules of C such that $\mathrm{Soc}\, C = \amalg S_\alpha$ in R-Mod. But then, since C is a cogenerator, the injective envelope of each simple left R-module must be isomorphic to one of the $E(S_\alpha)$, and so every injective left R-module must be gradable.

Most of the remaining results on graded artin algebras of Gordon and Green's [**4** and **5**] can now be shown to hold for graded left artinian rings, with only minor adjustments in their arguments. (For example, injectives in R-Gr are injective as R-modules.) However, their ultimate theorems [**5**, §§3, 4], showing that every module is gradable over a graded artin algebra of finite type, and that a graded artin algebra has finite type if and only if its category of gradable modules does, have proofs relying heavily on the Auslander-Reiten "dual of the transpose" which is defined only for artin algebras; there are no alternative arguments apparent to us.

We conclude with some applications relating the structure of R to that of R_0. If R is a semilocal graded ring, then by Proposition 4 the grading $R/J = \bigoplus R_n + J/J$ is finite and $R_0/J \cap R_0 \cong R_0 + J/J$ is semisimple. It follows from the latter that $J \cap R_0 = J_0$ (since always $J \cap R_0 \subseteq J_0$). Thus

8. PROPOSITION. *If R is semilocal, semiprimary, left perfect, or semiperfect with nil radical, then so is R_0 and R/J is finitely graded.*

On the other hand it follows from Theorem 1 and [**1**, Theorem 27.6] that

9. PROPOSITION. *If R_0 is semiperfect and R/J is finitely graded then R is semiperfect and has a complete set of primitive idempotents of degree zero; and so every projective R-module is gradable.*

We do not know whether R semiperfect implies R_0 semiperfect, but from the above propositions and minor modifications of the nilpotency argument of [**4**, Theorem 3.1] one can deduce that

10. PROPOSITION. *A finitely graded ring is semiprimary or left or right perfect if and only if its initial subring is.*

This is not the case for artinian rings as the ring of real matrices

$$\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{Q} \end{bmatrix}$$

clearly shows.

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