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ON THE MINIMAL EIGENVALUE OF THE LAPLACIAN OPERATOR FOR p-FORMS IN CONFORMALLY FLAT RIEMANNIAN MANIFOLDS[1]

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ABSTRACT. Let (M,g) be a compact orientable conformally flat Riemannian manifold and ${}^p\lambda_1$ the minimal eigenvalue of the Laplacian operator for p-forms. We prove that if there exists a positive constant K such that $\rho \geq Kg$, where ρ is the Ricci tensor of M, then ${}^p\lambda_1 \geq Kp(n-p+1)/(n-1)$ for each p, $1 \leq p \leq n/2$, $(n = \dim M)$; moreover if the equality holds for some p then M is of constant curvature $\sigma = K/(n-1)$.

1. Introduction. Let (M,g) be a compact orientable Riemannian manifold (connected, C^∞ and $\dim M = n$) with metric tensor g. By $\Lambda^p(M)$, $0 \leq p \leq n$, we denote the space of the p-forms and by ${}^p\Delta$ we denote the associated Laplace-Beltrami operator. It is given by

$$ {}^p\Delta = d\delta + \delta d \colon \Lambda^p(M) \to \Lambda^p(M) $$

where $d\colon \Lambda^p(M) \to \Lambda^{p+1}(M)$ is the exterior differential operator and $\delta\colon \Lambda^{p+1}(M) \to \Lambda^p(M)$ is the exterior codifferential operator.

The spectrum of ${}^p\Delta$, denoted by ${}^p\mathrm{Spec}(M,g)$, has the following form: $\{0 < {}^p\lambda_1 \cdots {}^p\lambda_1 < {}^p\lambda_2 \cdots {}^p\lambda_2 < {}^p\lambda_3 \cdots\} \to +\infty$ where each eigenvalue of ${}^p\Delta$ is repeated as many times as its multiplicity.

For (S^n, g_0), the sphere with the usual metric of positive constant sectional curvature $\sigma = K/(n-1)$, we have (E. Calabi):

$$ {}^0\lambda_1 = Kn/(n-1) \text{ and } {}^p\lambda_1 = Kp(n-p+1)/(n-1) \quad \text{for every } 1 \leq p \leq n/2. $$

A theorem of A. Lichnerowicz [**1**, p. 179] states what follows: if there exists a positive constant K such that $\rho \geq Kg$, where ρ denotes the Ricci tensor of M, then ${}^0\lambda_1 \geq Kn/(n-1)$.

Under the same hypothesis of Lichnerowicz's theorem, Obata [**1**, p. 180] has proved that if ${}^0\lambda_1 = Kn/(n-1)$, then (M,g) is isometric to (S^n, g_0).

In [**2**] S. Gallot and D. Meyer have obtained, under the hypothesis that the curvature operator S satisfies $S \geq K$ with K positive constant, the lower bound for ${}^p\lambda_1$ and have discussed the case when ${}^p\lambda_1$ takes the possible minimal value. For further topics related to estimates for eigenvalues of Δ see [**6**].

Now we assume our manifold to be conformally flat. Some results on the influence of the spectrum ${}^p\mathrm{Spec}(M,g)$, $p = 0, 1$, on the geometry of the conformally flat manifold have been studied in [**5**].

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In this paper we prove that if there exists a positive constant K such that $\rho \geq Kg$, then ${}^p\lambda_1 \geq Kp(n-p+1)/(n-1)$ for each p, $1 \leq p \leq n/2$, (cf. Theorem 3.2); moreover if the equality holds for some p then (M, g) is of constant curvature $\sigma = K/(n-1)$ (cf. Theorem 3.3).

2. Preliminaries. In the remainder of this paper, we denote by $g_{ij}, R^i_{jkh}, \rho_{ij} = R^r_{irj}$ the components of the metric, the curvature and the Ricci tensors respectively. By C^i_{jkh} we denote the components of the Weyl conformal curvature tensor. We shall represent tensors by their components with respect to the natural base, and shall use the summation convention. ∇ means the operator of covariant differentiation (with respect to the Riemannian connection) and $\tau = g^{ij}\rho_{ij}$ denotes the scalar curvature. For $\varphi, \psi \in \Lambda^p(M)$, the inner product (φ, ψ), the lengths $||\varphi||$ and $||\nabla\varphi||$ are given by

$$(\varphi, \psi) = \frac{1}{p!}\varphi_{i_1\cdots i_p}\psi^{i_1\cdots i_p}, \qquad ||\varphi||^2 = (\varphi, \varphi),$$
$$||\nabla\varphi||^2 = \frac{1}{p!}\nabla_r\varphi_{i_1\cdots i_p}\nabla^r\varphi^{i_1\cdots i_p}.$$

In §3 we shall use the following

LEMMA 2.1 [2, p. 270]. *For every p-form in an n-dimensional Riemannian manifold, we have*

$$||\nabla\varphi||^2 \geq \frac{1}{p+1}||d\varphi||^2 + \frac{1}{n-p+1}||\delta\varphi||^2.$$

3. The lower bound for ${}^p\lambda_1$. In this section we prove the main results.

THEOREM 3.1. *Let (M, g) be a compact orientable Riemannian manifold of dimension $n \geq 4$. If there exists a constant $K > 0$ such that $\rho_{ij} \geq Kg_{ij}$, then*

$${}^p\lambda_1 \geq \frac{Kp(n-p+1)}{n-1} - \frac{Hp(p-1)(n-p+1)}{2(n-p)} \quad \textit{for each } p \textit{ with } 1 \leq p \leq \frac{n}{2},$$

where we put

$$H = \mathrm{Sup}\left\{\frac{|C_{ijkh}\varphi^{ij}\varphi^{kh}|}{\varphi^{ij}\varphi_{ij}} : \varphi \in \Lambda^2(M)\right\}.$$

PROOF. For any $\varphi \in \Lambda^p(M)$ the following formula (cf. [4, p. 3]) holds:

$$\tfrac{1}{2}{}^0\Delta(||\varphi||^2) = (\varphi, {}^p\Delta\varphi) - ||\nabla\varphi||^2 - Q_p(\varphi) \tag{3.1}$$

where $Q_p(\varphi)$ is the quadratic form defined by

$$Q_p(\varphi) = \frac{1}{(p-1)!}\left[\rho_{ij}\varphi^{ii_2\cdots i_p}\varphi^{ji_2\cdots i_p} - \frac{p-1}{2}R_{ijkh}\varphi^{iji_3\cdots i_p}\varphi^{khi_3\cdots i_p}\right].$$

Since the Weyl's conformal curvature tensor is given by

$$R_{ijkh} = C_{ijkh} + \frac{1}{n-2}(\rho_{jh}g_{ik} + \rho_{ik}g_{jh} - \rho_{jk}g_{ih} - \rho_{ih}g_{jk})$$
$$- \frac{\tau}{(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk}),$$

we have the following formula:

$$R_{ijkh}\varphi^{iji_3\cdots i_p}\varphi^{kh}_{i_3\cdots i_p} = \frac{4}{n-2}\rho_{ij}\varphi^{ii_2\cdots i_p}\varphi^{j}_{i_2\cdots i_p} - \frac{2\tau}{(n-1)(n-2)}\varphi^{i_1\cdots i_p}\varphi_{i_1\cdots i_p} + C_{ijkh}\varphi^{iji_3\cdots i_p}\varphi^{kh}_{i_3\cdots i_p}.$$

Hence

$$\text{(3.2)}\quad Q_p(\varphi) = \frac{1}{(p-1)!}\left[\frac{n-2p}{n-2}\rho_{ij}\varphi^{ii_2\cdots i_p}\varphi^{j}_{i_2\cdots i_p} + \frac{(p-1)\tau}{(n-1)(n-2)}\varphi^{i_1\cdots i_p}\varphi_{i_1\cdots i_p} - \frac{p-1}{2}C_{ijkh}\varphi^{iji_3\cdots i_p}\varphi^{kh}_{i_3\cdots i_p}\right].$$

On the other hand, using the hypothesis of the theorem, fixing a point in the manifold and taking a coordinate system, we have

$$\rho_{ij}\varphi^{ii_2\cdots i_p}\varphi^{j}_{i_2\cdots i_p} \geq Kg_{ij}\varphi^{ii_2\cdots i_p}\varphi^{j}_{i_2\cdots i_p} = p!K\|\varphi\|^2, \qquad \tau = g^{ij}\rho_{ij} \geq nK,$$

$$-C_{ijkh}\varphi^{iji_3\cdots i_p}\varphi^{kh}_{i_3\cdots i_p} \geq -H\varphi^{iji_3\cdots i_p}\varphi_{iji_3\cdots i_p} = -p!H\|\varphi\|^2.$$

Consequently, from (3.2), we have

$$\text{(3.3)}\quad \begin{aligned} Q_p(\varphi) &\geq p\left[\frac{n-2p}{n-2}K + \frac{n(p-1)}{(n-1)(n-2)}K - \frac{p-1}{2}H\right]\|\varphi\|^2 \\ &= p\left[\frac{n-p}{n-1}K - \frac{p-1}{2}H\right]\|\varphi\|^2 \end{aligned}$$

at the fixed point in this coordinate system. But it is evident that this is valid also at all points and in all coordinate systems.

Now if φ is an eigenform for ${}^p\Delta$ corresponding to the eigenvalue ${}^p\lambda_1$, i.e. ${}^p\Delta\varphi = {}^p\lambda_1\varphi$ with $\|\varphi\|^2 \neq 0$, by integration of (3.1) over M with respect to the volume element dv we have

$$\text{(3.4)}\quad {}^p\lambda_1\int_M \|\varphi\|^2\,dv = \int_M \|\nabla\varphi^2\|\,dv + \int_M Q_p(\varphi)\,dv.$$

Moreover by integration of the inequality in Lemma 2.1, since $p \leq n-p$, we have

$$\text{(3.5)}\quad \begin{aligned} \int_M \|\nabla\varphi\|\,dv &\geq \frac{1}{n-p+1}\int_M (\|d\varphi\|^2 + \|\delta\varphi\|^2)\,dv \\ &= \frac{1}{n-p+1}\int_M (\varphi, {}^p\Delta\varphi)\,dv = \frac{{}^p\lambda_1}{n-p+1}\int_M \|\varphi\|^2\,dv. \end{aligned}$$

Finally by (3.4), because of (3.3) and (3.5), it follows that

$${}^p\lambda_1\frac{n-p}{n-p+1}\int_M \|\varphi\|^2\,dv \geq p\left[\frac{n-p}{n-1}K - \frac{p-1}{2}H\right]\int_M \|\varphi\|^2\,dv.$$

This completes the proof of Theorem 3.1.

THEOREM 3.2. *Let (M,g) be a compact orientable conformally flat Riemannian manifold of dimension $n \geq 4$. If there exists a constant $K > 0$ such that $\rho_{ij} \geq K g_{ij}$, then*

$$^{p}\lambda_1 \geq \frac{p(n-p+1)}{(n-1)} K \quad \textit{for each } p, 1 \leq p \leq \frac{n}{2}.$$

PROOF. This follows from Theorem 3.1 since for a conformally flat Riemannian manifold the tensor C_{ijkh} is a zero tensor.

REMARK 3.1. For $p = 1$ Theorem 3.2 holds also without the hypothesis that (M,g) be conformally flat.

REMARK 3.2. The Hodge star operator $*$ defines an isomorphism $\Lambda^p(M) \to \Lambda^{n-p}(M)$; besides if φ is a p-form such that $^{p}\Delta\varphi = \lambda\varphi$ then $*\varphi$ is a $(n-p)$-form such that $^{n-p}\Delta(*\varphi) = \lambda(*\varphi)$.

Consequently $^{p}\lambda_1 = {}^{n-p}\lambda_1$, and thus from Theorem 3.2 we have

$$^{p}\lambda_1 = {}^{n-p}\lambda_1 = \frac{(n-p)(p+1)}{(n-1)} K \quad \text{for every } p, \frac{n}{2} \leq p \leq n-1.$$

Now we prove a theorem analogous to the theorem of Obata.

THEOREM 3.3. *In the same hypothesis of Theorem 3.2, if*

$$^{p}\lambda_1 = \frac{p(n-p+1)}{n-1} K \quad \textit{for some } p, 1 \leq p \leq \frac{n}{2},$$

or

$$^{p}\lambda_1 = \frac{(n-p)(p+1)}{(n-1)} K \quad \textit{for some } p, \frac{n}{2} \leq p \leq n-1,$$

then (M,g) is a Riemannian manifold of constant sectional curvature $\sigma = K/(n-1)$.

PROOF. We denote by E_λ the eigenspace of p-forms corresponding to λ. If we put

$$^{p}\lambda_1' = \inf\{\lambda / E_\lambda' \neq \{0\}\} \quad \text{and} \quad ^{p}\lambda_1'' = \inf\{\lambda / E_\lambda'' \neq \{0\}\},$$

where

$$E_\lambda' = \{\varphi \in E_\lambda / d\varphi = 0\} \quad \text{and} \quad E_\lambda'' = \{\varphi \in E_\lambda / \delta\varphi = 0\},$$

by a proof similar to that of Theorem 3.1, it is easy to see that

$$^{p}\lambda_1' \geq \frac{p(n-p+1)}{n-1} K \quad \text{and} \quad ^{p}\lambda_1'' \geq \frac{(p+1)(n-p)}{n-1} K \qquad (1 \leq p \leq n).$$

Moreover for the decomposition of Hodge-deRham we have $E_\lambda = E_\lambda' \oplus E_\lambda''$ and hence $^{p}\lambda_1 = \inf\{^{p}\lambda_1', {}^{p}\lambda_1''\}$.

On the other hand

$$^{p}\lambda_1 \geq \frac{p(n-p+1)}{n-1} K \quad \text{for } 1 \leq p \leq \frac{n}{2}$$

and

$$^{p}\lambda_1 \geq \frac{(p+1)(n-p)}{n-1} K \quad \text{for } \frac{n}{2} \leq p \leq n-1.$$

Therefore $^{p}\lambda_1 = {}^{p}\lambda_1'$ for $1 \leq p \leq n/2$, and $^{p}\lambda_1 = {}^{p}\lambda_1''$ for $n/2 \leq p \leq n-1$. Thus if we consider an eigenform φ corresponding to $^{p}\lambda_1 = (p(n-p+1)/(n-1))K$, $1 \leq p \leq n/2$, then it is closed i.e. $d\varphi = 0$.

In particular for $p = 1$ we have

$$^0\Delta\delta\varphi = \delta^1\Delta\varphi = \frac{nK}{n-1}\delta\varphi \quad \text{with}\ \delta\varphi \neq 0,$$

i.e. $\delta\varphi$ is an eigenfunction associated to the eigenvalue $nK/(n-1)$. From Obata's theorem it follows that M is of constant sectional curvature $\sigma = K/(n-1)$.

Now we suppose $1 < p \leq n/2$. Because ${}^p\lambda_1 = (p(n-p+1)/(n-1))K$, (3.4) becomes

$$\int_M \left(\|\nabla\varphi\|^2 - \frac{pK}{n-1}\|\varphi\|^2\right) dv + \int_M \left(Q_p(\varphi) - \frac{p(n-p)}{n-1}K\|\varphi\|^2\right) dv = 0. \tag{3.6}$$

On the other hand

$$\|\nabla\varphi\|^2 \geq \frac{1}{n-p+1}\|\delta\varphi\|^2 = \frac{1}{n-p+1}(\varphi, {}^p\Delta\varphi) = \frac{pK}{n-1}\|\varphi\|^2$$

and

$$\begin{aligned} Q_p(\varphi) &= \frac{n-2p}{(p-1)!(n-2)}\rho_{ij}\varphi^{ii_2\cdots i_p}\varphi^j_{i_2\cdots i_p} + \frac{p(p-1)\tau}{(n-1)(n-2)}\|\varphi\|^2 \\ &\geq \frac{p(n-2p)}{n-2}K\|\varphi\|^2 + \frac{np(p-1)}{(n-1)(n-2)}K\|\varphi\|^2 \\ &= \frac{p(n-p)}{n-1}K\|\varphi\|^2. \end{aligned}$$

Therefore, from (3.6), we have

$$\|\nabla\varphi\|^2 = \frac{1}{n-p+1}\|\delta\varphi\|^2 \quad \text{and} \quad Q_p(\varphi) = \frac{p(n-p)}{n-1}K\|\varphi\|^2.$$

In particular we have

$$p(p-1)(\tau - nK)\|\varphi\|^2 = 0 \quad \text{with}\ p > 1\ \text{and}\ \|\varphi\|^2 \neq 0,$$

for which $\tau = nK = \text{const}$.

Summing up we have that M is a compact orientable conformally flat Riemannian manifold with constant scalar curvature and with positive Ricci curvature. Using Theorem A of Tani [7], we have that M is of constant sectional curvature $\sigma = K/(n-1)$ (in fact $n(n-1)\sigma = \tau = nK$).

For Remark 3.2, Theorem 3.3 holds also if ${}^p\lambda_1 = (n-p)(p+1)K/(n-1)$ for some p, $n/2 \leq p \leq n-1$. Q.E.D.

The following corollary is a consequence of Theorem 3.3 and of Theorem 7.10 of [3, p. 265].

COROLLARY 3.1. *In the same hypothesis of Theorem* 3.3, *if M is simply connected, then it is isometric to the sphere (S^n, g_0) of positive constant sectional curvature $K/(n-1)$.*

REMARK 3.3. For manifolds of dimension $n = 3$, that are not conformally flat, Theorem 3.2 is again valid since for every 3-dimensional Riemannian manifold the tensor C_{ijkh} is a zero tensor. In dimension 3 also Theorem 3.3 is valid without the hypothesis that M be conformally flat. In fact for $p = 1$ (as seen already) Theorem 3.3 follows from Obata's theorem and, for $p = 2$, ${}^2\lambda_1 = {}^{n-1}\lambda_1 = {}^1\lambda_1$.

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