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# EVERY TWO-GENERATOR KNOT IS PRIME[1]

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ABSTRACT.
THEOREM. *Every two-generator knot is prime.*
The proof gives conditions under which a free product with amalgamation cannot be generated by two elements. An example is given of a composite one-relator link.

Let $k$ be a tame knot in $S^3$. Then the deficiency of $\pi_1(S^3 - k)$ is one. That is to say, $\pi_1(S^3 - k)$ has a presentation in which the number of generators is one more than the number of relators, but no presentation in which the number of generators is two more than the number of relators. Thus, every one-relator knot is a two generator knot. We prove that every two-generator knot is prime by showing that a composite-knot group cannot be generated by any two of its elements.

The most common examples of one-relator knots are torus knots and two-bridge knots. The knot $10_{132}$ (Rolfsen's table, [3]) is a one-relator knot which is neither a torus knot nor a two-bridge knot. The knot $9_{46}$ is a prime knot which is not a one-relator knot. The link $2_1^2 \# 3_1$ is a composite, one-relator link!

The group of a composite knot can be written as a free product amalgamated over **Z**. In 1942, F. W. Levi proved that the minimum number of generators of a free product is the sum of the corresponding numbers of the free factors, provided this sum is 2 or infinity. In a remarkable 1943 paper [1], B. H. Neumann proved that this is true in all other cases as well. Unfortunately, no such simple result is possible for free products with amalgamation, as the example $\pi_1(S^3 - 2_2^2 \# 3_1)$ shows. In this example, the free product of two two-generator groups amalgamated over **Z**, yields a two-generator, one-relator group. By contrast, it is easy to show that $\pi_1(S^3 - 3_1 \# 3_1)$, a free product of two two-generator groups amalgamated over **Z**, yields a three-generator, two-relator group.

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**1. Preliminaries.** We are in the PL-category; in particular, all knots are tame. In our proof that the group of a composite knot cannot be a two-generator group, we

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[1]This theorem was proved as part of the author's 1979 Ph. D. thesis [2], directed by Wilbur Whitten at the University of Southwestern Louisiana. The research was partially supported by NSF grant no. MCS 77-18723A04.

make frequent use of the fact that if $x$ and $y$ generate a group $G$, and if $g$ is an element of $G$, then $x^g$ and $y^g$ generate $G$.

We begin by briefly summarizing some properties of a free product with amalgamation [4] needed in the sequel. Let $A *_C B$ be a free product with amalgamation. Then any element of $A *_C B$ can be written in normal form $x_1 x_2 \cdots x_n c$, where $c \in C$ and the $x_i$'s are alternately from $A$ and $B$ and are nontrivial elements of left transversals for $C$ in $A$ and in $B$. If $x$ in $A *_C B$ has normal form $x = x_1 x_2 \cdots x_n c$, then the length of $x$, written $\lambda(x)$ is $n$. If $x_1 \in A$, we say $x$ begins in $A$ and write $x = a_1 b_2 a_3 b_4 \cdots$, where $x_1 = a_1$, $x_2 = b_2$, etc. If $x_n \in A$, we say $x$ ends in $A$ and write $x = \cdots b_{n-1} a_n c$, where $x_{n-1} = b_{n-1}$, $x_n = a_n$. Similarly, $x$ may begin or end in $B$. If we write $x = a_1 b_2 \cdots a_{n-1} b_n c$, we mean to imply the following: that $x$ begins in $A$, that $x$ ends in $B$, and that the length of $x$ is $n$.

To multiply two elements $x$ and $y$ in $A *_C B$, we concatenate their normal forms and then cancel and reduce the resulting word, moving elements of $C$ to the end of the word, until we obtain a normal form for the product. Suppose $x = a_1 b_2 a_3$ and $y = a_1' b_2' a_3'$. Then to find $xy$ we first write $a_1 b_2 a_3 a_1' b_2' a_3'$. If $a_3 a_1' \notin C$, then a normal form for $xy$ is $x_1 x_2 x_3 x_4 x_5$, where $x_1 = a_1$, $x_2 = b_2$, $x_3 = a_3 a_1'$, $x_4 = b_2'$ and $x_5 = a_3'$. We call this procedure of combining $a_3 a_1'$ into a single symbol, *amalgamation*. If, on the other hand, $a_3 a_1'$ is in $C$, then we have *cancellation*, and must move $a_3 a_1' = c'$ to the end of the word.

If $x$ ends in $A$ and $y$ begins in $B$, then when we form $xy$ we have neither amalgamation nor cancellation, and $\lambda(xy) = \lambda(x) + \lambda(y)$.

Let $x = x_1 \cdots x_n c$ and $y = y_1 \cdots y_m c'$. Suppose $x_{n-j+1} \cdots x_n c y_1 \cdots y_j \in C$ but $x_{n-j} x_{n-j+1} \cdots x_n c y_1 \cdots y_j y_{j+1} \notin C$. Then we say that when we multiply $x$ and $y$, exactly $j$ letters cancel. Unless the number of letters which cancel is equal to $\lambda(x)$ or $\lambda(y)$, the cancellation is always followed by an amalgamation. Therefore, if $j$ letters cancel in $xy$, and $j \neq \lambda(x)$, $j \neq \lambda(y)$, then $\lambda(xy) = \lambda(x) + \lambda(y) - 2j - 1$. If $j = \lambda(x)$, then $\lambda(xy) = \lambda(y) - \lambda(x)$. If $j = \lambda(y)$, then $\lambda(xy) = \lambda(x) - \lambda(y)$.

We say that $A *_C B$ is a *nontrivial free product with amalgamation* iff neither of the injections $C \hookrightarrow A$ and $C \hookrightarrow B$ is a surjection.

**2. Two-generator knots are prime.** Stallings' Theorem 4.3 in [6] implies the following:

LEMMA 1. *If $n$ elements generate a free product with amalgamation then there exists a set of $n$ generators one of which is an element of one of the free factors.*

LEMMA 2. *Let $A *_C B$ be nontrivial, with $C \neq 1$. Suppose that for each $x$ in $A *_C B$ with $x$ not in $C$, if for some $n$ in $\mathbf{Z}$, $x^n$ is in $C$, then in fact, $x^n = 1$. If $A *_C B$ can be generated by two elements, then there exists a generating pair with normal forms $g_1 = a_1 b_2 \cdots b_n c$ and $g_2 = c'$.*

There are seven possible normal forms for a word in $A *_C B$: $a_1 b_2 \cdots a_n c$, $b_1 a_2 \cdots b_n c$, $a_1 b_2 \cdots b_n c$, $b_1 a_2 \cdots a_n c$, $ac$, $bc$, and $c$ ($n \geqslant 2$). Lemma 1 says that if two elements generate $A *_C B$ we can find a pair of generators one of which has one of the last three forms on this list.

Also, we need never use form number four. If one generator has this form, we may replace it by its inverse, which has the same length and has form number three. And we need not consider pairs of generators both of which lie in $A$ or both in $B$, if we assume $A *_C B$ is nontrivial. This leaves the following cases to consider:

(1a) $\{a_1b_2 \cdots a_nc, a'c'\}$,
(1b) $\{b_1a_2 \cdots b_nc, b'c'\}$,
(2a) $\{a_1b_2 \cdots a_nc, b'c'\}$,
(2b) $\{b_1a_2 \cdots b_nc, a'c'\}$,
(3a) $\{a_1b_2 \cdots a_nc, c'\}$,
(3b) $\{b_1a_2 \cdots b_nc, c'\}$,
(4) $\{a_1b_2 \cdots b_nc, a'c'\}$,
(5) $\{a_1b_2 \cdots b_nc, b'c'\}$,
(6) $\{a_1b_2 \cdots b_nc, c'\}$,
(7) $\{ac, b'c'\}$.

We reduce cases 1 and 3 to cases 4, 5 and 6 by conjugation. For example, if we are given generators with normal form (1a), we conjugate by $a_1$ to obtain generators with normal form 4 (after taking the inverse of the first generator.)

Pairs of elements with normal forms 2 or 7 cannot generate. For example, suppose $g_1$ and $g_2$ have form (2a). Then the length of $g_1$ is odd, so we can write $g_1 = a_1b_2 \cdots a_nc = fmtc$, where for $j = \frac{1}{2}(n-1)$ we have $f = x_1x_2 \cdots x_j$, $m = x_{j+1}$, and $t = x_{j+2} \cdots x_n$. Now, $g_1$ begins and ends in $A$. Can we have a power of $g_1$ which does not begin and end in $A$? Suppose for some natural number $e$ we have $g_1^e$ does not begin and end in $A$. Then we must have cancellation, so $tcf = k$, an element of $C$, and $g_1^e = f(mk)^{e-1}mtc$. We must have further cancellation, so $(mk)^{e-1}m = K$, an element of $C$, and the last letter of $f$ must cancel the first letter of $t$ and so on. (Note that cancellation is the same whether we multiply before or after we commute $K$ to the end of the word.)

But $f$ and $t$ have the same length, and letters cancel in pairs, so if $g_1^e$ does not begin in $A$, then $g_1^e$ is in $C$ and so, by hypothesis, equals 1. A similar argument goes through for negative powers of $g_1$. We have shown that every power of $g_1$ either begins and ends in $A$ or else equals 1.

Thus when we form words in $g_1$ and $g_2$ we get no cancellation, because the powers of $g_1$ begin and end in $A$ while the powers of $g_2$ begin and end in $B$. Words in $g_1$ and $g_2$ can never equal a nontrivial element of $C$. By hypothesis, $C \neq 1$. Therefore $g_1$ and $g_2$ cannot generate.

Cases 4 and 5 also reduce to case 6. For example, suppose $g_1$ and $g_2$ are generators with normal form (4), so $g_1 = a_1b_2 \cdots b_nc$ and $g_2 = a'c'$. Since words in these generators must equal nontrivial elements of $C$, in some such word $b_2$ must cancel, and before this can happen $a_1$ must cancel. Therefore, for some integer $e$, we have $a_1^{-1}(a'c')^ea_1 = (a_1^{-1}a'c'a_1)^e$ is an element of $C$. Because we do not want $(a'c')^e$ to be 1, this element must not be 1, and so, by hypothesis, $a_1^{-1}a'c'a_1$ must be in $C$. Therefore, if we conjugate $g_1$ and $g_2$ by $a_1$ and then take the inverse of the first new generator, we obtain new generators of form (6). Case 5 is similar.

This proves the lemma.

(We can also show that there is a pair of generators of normal form (6) with the property that the sum of their lengths is minimal among all pairs of generators.)

Case 6 is the one which actually occurs in the example $\pi_1(S^3 - 2_1^2 \# 3_1) = | a, b, c: aca = cac, bc = cb |$. Let $g_1 = ab$ and $g_2 = c$. Then $g_2 g_1 g_2^{-1} g_1^{-1} g_2^{-1} g_1 = b$.

LEMMA 3. *Let $A *_C B$ obey the hypotheses of Lemma 2, and the additional hypothesis that there is some a in A such that for all c in C, $C \neq 1$, $a^{-1}ca$ is not in C, and there is some b in B with this same property. Then $A *_C B$ is not generated by any two of its elements.*

Assume, by way of contradiction, that $g_1$ and $g_2$ generate $A *_C B$. By Lemma 2, we can take $g_1 = a_1 b_2 \cdots b_n c$ and $g_2 = c'$.

For any element $a$ of $A$ there is a word in $g_1$ and $g_2$ which equals $a$. In this word, all of the $b_i$'s must cancel, and so the $a_i$'s and $C$ must generate $A$. Similarly, the $b_i$'s and $C$ must generate $B$. Therefore there is some $a_i$ and some $b_j$ such that we never get an element of $C$ when we conjugate a nontrivial element of $C$ by either.

Therefore, when we form the words $g_1^x g_2^y g_1^z$, $x, y, z$ integers, $a_i$ and $b_j$ do not cancel, and we do not get a word of length 1. If all the letters of $g_1$ canceled except one, we might still get cancellation when we formed longer words, such as $g_1^{-1} g_2^x g_1 g_2^y g_1^{-1} g_2^z$, but since two letters of $g_1$ never cancel, words such as these also never have length 1. Therefore $g_1$ and $g_2$ do not generate. This contradiction proves the lemma.

THEOREM. *Every two-generator knot is prime.*

The group of a composite knot is a free product amalgamated over **Z**, so we need only show that knot groups satisfy the hypotheses of Lemma 3.

Let $k_1$ and $k_2$ be nontrivial knots, with fundamental groups $A$ and $B$. Let $C$ be a meridianal subgroup of $A$ and let $C'$ be a meridianal subgroup of $B$. Then $C$ and $C'$ are both isomorphic to **Z**, and the fundamental group of $S^3 - k_1 \# k_2$ is $A *_C B$.

This free product with amalgamation is nontrivial and $C$ is not 1. We now use a theorem of J. Simon [**5**], which says that unless a knot is a cable knot no power of an element not in its peripheral subgroup can be a nontrivial element of its peripheral subgroup. Since the meridianal subgroup is a direct summand of the peripheral subgroup, this property extends to meridianal subgroups. Cable knots are prime and $k_1 \# k_2$ is composite, so this property holds for $A *_C B$.

Given this property and the fact that $C = \mathbf{Z}$, the additional hypothesis of Lemma 3 is equivalent to the assumption that $C$ is not normal in $A$ and $C$ is not normal in $B$. The normal closure of a meridianal subgroup in a knot group is the entire group, so we have this property as well.

Therefore every two-generator knot is prime.

## REFERENCES

1. B. H. Neumann, *On the number of generators of a free product*, J. London Math. Soc. **18** (1943), 12–20.
2. F. H. Norwood, *One-relator knots*, Dissertation, Univ. Southwestern Louisiana, 1979.
3. D. Rolfsen, *Knots and links*, Math. Lectures, vol. 7, Publish or Perish, Berkeley, Calif., 1976.
4. J. J. Rotman, *The theory of groups: An introduction*, Allyn and Bacon, Boston, Mass., 1965.

5. J. Simon, *Roots and centralizers of peripheral elements in knot groups*, Math. Ann. **222** (1976), 205–209.

6. J. Stallings, *A topological proof of Grushko's theorem on free products*, Math. Z. **90** (1965), 1–8.

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