PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 1, September 1982

# DIFFERENTIABILITY OF CONVEX FUNCTIONS AND RYBAKOV'S THEOREM

RUSSELL G. BILYEU AND PAUL W. LEWIS

ABSTRACT. Rybakov proved that if $\mu: \Sigma \to X$ is a countably additive Banach space valued measure on a $\sigma$-algebra $\Sigma$, then there is an element $x^* \in X^*$ so that $\mu \ll x^*\mu$. In this note, we show that Rybakov's theorem follows essentially from a classical result of Mazur on the Gateaux differentiability of convex functions.

Suppose that $(\Omega, \Sigma)$ is a measurable space, $X$ is a (real) Banach space with dual $X^*$, and $\mu: \Sigma \to X$ is a countably additive vector valued measure. Let ca$(\Sigma)$ denote the Banach space ($\|\nu\| = |\nu|(\Omega)$) of all countably additive real valued measures defined on $\Sigma$. If $\nu, \lambda \in$ ca$(\Sigma)$, then $D(\nu, \lambda)$ will denote $\lim_{t\to 0}(1/t)(\|\nu + t\lambda\| - \|\nu\|)$, provided this limit exists. The one-sided limits will be denoted by $D^+(\nu, \lambda)$ and $D^-(\nu, \lambda)$. Since the norm is a convex function, $D^+(\nu, \lambda) \geq D^-(\nu, \lambda)$; also, $D(\nu, \lambda)$ exists iff $D^+(\nu, \lambda) = -D^+(\nu, -\lambda) = D^-(\nu, \lambda)$.

We begin our demonstration of Rybakov's theorem by showing that if $D(\nu, \lambda)$ exists, then $\lambda \ll \nu$. Write $\lambda = \lambda_a + \lambda_s$, where $\lambda_a \ll |\nu|$ and $|\lambda_s| \perp |\nu|$. Then

$$\begin{aligned} D^+(\nu, \lambda) &= \lim_{t\to 0+}(1/t)(\|\nu + t\lambda\| - \|\nu\|) \\ &= \lim_{t\to 0+}(1/t)(\|\nu + t\lambda_a\| - \|\nu\| + \|t\lambda_s\|) = D^+(\nu, \lambda_a) + \|\lambda_s\|. \end{aligned}$$

Similarly, $D^-(\nu, \lambda) = \lim_{t\to 0-}(1/t)(\|\nu + t\lambda\| - \|\nu\|) = D^-(\nu, \lambda_a) - \|\lambda_s\|$. If $D(\nu, \lambda)$ exists, then $0 \leq D^+(\nu, \lambda_a) - D^-(\nu, \lambda_a) = -2\|\lambda_s\|$. Consequently, $\|\lambda_s\| = 0$ and $\lambda \ll \mu$.

Next, let $(x_n^*)$ be a sequence of points in the unit ball of $X^*$ so that $\mu \ll \sum_{n=1}^{\infty}(1/2^n)|x_n^*\mu|$ [**1**; **2**, p. 12; **3**]. Modifying the outline of Mazur's theorem in [**4**, p. 450], for $y^* \in X^*$ and $m \in \mathbf{N}$, put $\mathcal{D}(y^*, m) = \{x^* \in X^*: D^+(x^*\mu, y^*\mu) + D^+(x^*\mu, -y^*\mu) < 1/m\}$. Now let $x_0^* \in X^*$ and set $g(t) = \|(x_0^* + ty^*)\mu\|$ for $t \in \mathbf{R}$. Thus $g$ is a continuous convex function, and consequently $g$ is differentiable at all but countably many points. But if $g$ is differentiable at $t_0$, then $D((x_0^* + t_0 y^*)\mu, y^*\mu)$ exists, i.e. $D^+((x_0^* + t_0 y^*)\mu, y^*\mu) = -D^+((x_0^* + t_0 y^*)\mu, -y^*\mu)$ and $x_0^* + t_0 y^* \in \mathcal{D}(y^*, m)$. Therefore $\mathcal{D}(y^*, m)$ is a dense subset of $X^*$ for each $y^* \in X^*$ and each $m \in \mathbf{N}$. Furthermore, the set

$$A_k = \{x^* \in X^*: k(\|x^*\mu + (1/k)y^*\mu\| - \|x^*\mu\| + \|x^*\mu - (1/k)x^*\mu\| - \|y^*\mu\|) < 1/m\}$$

is certainly open for all positive integers $k$ and $m$. And because of the monotonicity of the difference quotients of a convex function, $\mathcal{D}(y^*, m) = \bigcup_{k=1} A_k$, a dense open set. Consequently, by the Baire category theorem, $\bigcap_{m=1}^{\infty} \mathcal{D}(y^*, m)$ is a dense $G_\delta$ for each $y^* \in X^*$, and $D(x^*\mu, y^*\mu)$ exists for each $x^*$ in this intersection. Hence $\mathcal{D} = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{D}(x_n^*, m)$ is a dense $G_\delta$, and $\mu \ll x^*\mu$ for each $x^* \in \mathcal{D}$.

Received by the editors February 20, 1982.
1980 *Mathematics Subject Classification*. Primary 46G10, 28B05; Secondary 46B20.

## REFERENCES

1. R. G. Bartle, N. Dunford and J. T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), 289–305.
2. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R. I., 1977.
3. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
4. S. Mazur, *Über konvexe Mengen in linearen normierte Räumen*, Studia Math. 4 (1933), 70–84.
5. V. Rybakov, *Theorem of Bartle, Dunford, and Schwartz on vector-valued measures*, Mat. Zametki 7 (1970), 247–254.

DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY, DENTON, TEXAS 76203