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L^1-CONVERGENCE OF FOURIER SERIES WITH COMPLEX QUASIMONOTONE COEFFICIENTS

VERA B. STANOJEVIC

ABSTRACT. A sequence of Fourier coefficients $\{\hat{f}(n)\}$ of a complex function in $L^1(T)$ is said to be complex quasimonotone if there exists θ_0 such that

$$\Delta \hat{f}(n) + \frac{\alpha}{n}\hat{f}(n) \in \left\{ z \mid |\arg z| \leq \theta_0 < \frac{\pi}{2} \right\}$$

for some $\alpha \geq 0$ and for all n. It is proved that Fourier series with asymptotically even and complex quasimonotone coefficients, satisfying

$$\overline{\lim_{n\to\infty}}\, n^{1/q} \max_{n \leq j \leq [\lambda n]} |\Delta \hat{f}(j)|^{1/q} \max_{n \leq j \leq [\lambda n]} |\hat{f}(j)|^{1/p} = o(1),$$

$$\lambda \to 1+0,\ \tfrac{1}{p} + \tfrac{1}{q} = 1,$$

converges in $L^1(T)$-norm if and only if $\hat{f}(n)\lg|n| = o(1)$, $n \to \infty$. A recent result of Č. V. Stanojević [3] is a special case of the corollary of the main theorem.

1. Introduction. Recently Č. V. Stanojević [1] introduced a Tauberian L^1-convergence class of Fourier series

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}. \tag{1.1}$$

The sequence of Fourier coefficients $\{\hat{f}(n)\}$ belongs to the class $\mathcal{HK}$ if for some $1 < p \leq 2$

$$\lim_{\lambda \to 1+0} \overline{\lim_{n\to\infty}} \left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} = 0. \tag{HK}$$

For $\{\hat{f}(n)\} \in \mathcal{HK}$ and $\{\hat{f}(n)\}$ asymptotically even, i.e.

$$\frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - \hat{f}(-k)| \lg k = o(1), \qquad n \to \infty, \tag{AE1}$$

$$\lim_{\lambda \to 1+0} \overline{\lim_{n\to\infty}} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j = 0, \tag{AE2}$$

it is shown in [1] that the Fourier series (1.1) converges in $L^1(T)$-norm, where $T = \mathbf{R}/2\pi\mathbf{Z}$, if and only if

$$\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\|_{L^1(T)} = o(1), \qquad n \to \infty, \tag{ST}$$

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where

$$E_n(t) = \sum_{k=0}^{n} e^{ikt}.$$

It was observed in [1] that $\{\hat{f}(n)\} \in \mathcal{HK}$ implies that

$$\text{(BOX)} \qquad n^{1/q}\Delta \hat{f}(n) = o(1), \qquad n \to \infty, \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Proposition 3.1 in [1] and further results of W. O. Bray and Č. V. Stanojević [2] indicate that a stronger form of (BOX) together with some additional assumptions about the speed of $\{\hat{f}(n)\}$ can be used to obtain new classes of L^1-convergence. However, it is natural to ask are there some reasonable regularity conditions for $\{\hat{f}(n)\}$ such that a slightly stronger form of (BOX) would imply that (ST) is a necessary and sufficient condition for L^1-convergence of (1.1).

In this paper we shall show that it is possible to define monotonicity for a sequence of complex numbers, apply that definition to $\{\hat{f}(n)\} \in \mathcal{HK}$ and obtain some (BOX)-like conditions. It will be also shown that a result of Č. V. Stanojević [3] (Corollary 2.2) is a special case of a corollary to our main result.

2. Definitions and lemmas. In this section we shall define quasimonotonicity of a sequence of complex numbers $\{c_n\}$ by restricting the range of the sequence $\{\Delta c_n + \alpha c_n/n\}$ to a certain cone in the complex plane.

DEFINITION 2.1. A sequence of complex numbers $\{c_n\}_{n=1}^{\infty}$ is complex quasimonotone if there exists a cone

$$K_\alpha(\theta_0) = \left\{ z \,|\, |\arg z| \leq \theta_0 < \frac{\pi}{2} \right\}$$

such that for some $\alpha \geq 0$

$$\Delta c_n + \frac{\alpha}{n} c_n \in K_\alpha(\theta_0)$$

for all n.

The following lemma gives an estimate of $\sum_{j=n}^{m} |\Delta c_j|$ that we need in our main result.

LEMMA 2.1. *Let $\{c_n\}_{n=1}^{\infty}$ be a complex quasimonotone sequence. Then*

$$\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{|c_m - c_n|}{\cos\theta_0} + \alpha\left(1 + \frac{1}{\cos\theta_0}\right) \sum_{j=n}^{m} \frac{|c_j|}{j}. \tag{2.1}$$

PROOF. To the first term of the right-hand side of the inequality

$$\sum_{j=n}^{m-1} |\Delta c_j| \leq \sum_{j=n}^{m-1} \left|\Delta c_j + \frac{\alpha}{j} c_j\right| + \alpha \sum_{j=n}^{m} \frac{|c_j|}{j},$$

we apply the inequality of M. Petrović [4], i.e.

$$\sum_{j=n}^{m-1} \left|\Delta c_j + \frac{\alpha}{j} c_j\right| \leq \frac{1}{\cos\theta_0} \left| \sum_{j=n}^{m-1} \left(\Delta c_j + \frac{\alpha}{j} c_j\right) \right|$$

and obtain

$$\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos\theta_0} \left| \sum_{j=n}^{m-1} \Delta c_j \right| + \frac{\alpha}{\cos\theta_0} \sum_{j=n}^{m} \frac{|c_j|}{j} + \alpha \sum_{j=n}^{m} \frac{|c_j|}{j}.$$

Finally

$$\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos\theta_0} |c_m - c_n| + \alpha\left(\frac{1}{\cos\theta_0} + 1 \right) \sum_{j=n}^{m} \frac{|c_j|}{j}.$$

In particular, for $\alpha = 0$, we have the following definition and the corresponding lemma.

DEFINITION 2.2. A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is complex monotone if there exists a cone

$$K(\theta_0) = \left\{ z \,|\, |\arg z| \leq \theta_0 < \frac{\pi}{2} \right\}$$

such that $\Delta c_n \in K(\theta_0)$, for every n.

LEMMA 2.2. *Let $\{c_n\}_{n=1}^{\infty}$ be a complex monotone sequence. Then*

$$\sum_{j=n}^{m-1} |\Delta c_j| \leq \frac{1}{\cos\theta_0} |c_m - c_n|. \tag{2.2}$$

From (2.2) it is clear that complex monotonically decreasing sequence $\{c_n\}$ is of bounded variation.

To state our result in a more succinct form we shall use the following lemma from [2].

LEMMA 2.3. *The condition* (ST) *is equivalent to*

$$\hat{f}(n) \lg|n| = o(1), \qquad n \to \infty.$$

3. Results. Our main result gives necessary and sufficient conditions for L^1-convergence of Fourier series with asymptotically even coefficients and satisfying some (BOX)-like conditions.

THEOREM 3.1. *Let*

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}$$

be the Fourier series of $f \in L^1(T)$ with asymptotically even coefficients.

If $\{\hat{f}(n)\}$ is complex quasimonotone and if for some $1 < p \leq 2$,

$$\overline{\lim_{n\to\infty}}\, n^{1/q} \max_{n \leq j \leq [\lambda n]} |\Delta \hat{f}(j)|^{1/q} \max_{n \leq j \leq [\lambda n]} |\hat{f}(j)|^{1/p} = o(1), \tag{3.1}$$

$$\lambda \to 1 + 0, \tfrac{1}{p} + \tfrac{1}{q} = 1,$$

then

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$\hat{f}(n)\lg|n| = o(1), \qquad n \to \infty.$$

PROOF. It suffices to prove that (3.1) and complex quasimonotonicity imply

$$\text{(HK)} \qquad \lim_{\lambda \to 1+0} \overline{\lim_{n\to\infty}} \left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} = 0.$$

From

$$\left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \leqslant \max_{n \leqslant j \leqslant [\lambda n]} \left(j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \left(\sum_{j=n}^{[\lambda n]} |\Delta \hat{f}(j)| \right)^{1/p},$$

using Lemma 2.1 we obtain

$$\begin{aligned}
&\left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \\
&\quad \leqslant \max_{n \leqslant j \leqslant [\lambda n]} \left(j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \left(\frac{|\hat{f}([\lambda n]) - \hat{f}(n)|}{\cos \theta_0} \right. \\
&\qquad\qquad \left. + \alpha \left(1 + \frac{1}{\cos \theta_0} \right) \sum_{j=n}^{[\lambda n]} \frac{|\hat{f}(j)|}{j} \right)^{1/p} \\
&\quad \leqslant \max_{n \leqslant j \leqslant [\lambda n]} \left(j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \left(\frac{|\hat{f}([\lambda n]) + \hat{f}(n)|}{\cos \theta_0} \right. \\
&\qquad\qquad \left. + \alpha \left(1 + \frac{1}{\cos \theta_0} \right) \max_{n \leqslant j \leqslant [\lambda n]} |\hat{f}(j)| \sum_{j=n}^{[\lambda n]} \frac{1}{j} \right)^{1/p}.
\end{aligned}$$

But both $|\hat{f}([\lambda n])|$ and $|\hat{f}(n)|$ are less than $\max_{n \leqslant j \leqslant [\lambda n]} |\hat{f}(j)|$, and $\sum_{j=n}^{[\lambda n]} \frac{1}{j} \leqslant C \lg \lambda$, where C is an absolute constant.

Hence, for $\lambda > 1$ we have

$$\begin{aligned}
\left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} &\leqslant \left[\frac{2}{\cos \theta_0} + \alpha \left(1 + \frac{1}{\cos \theta_0} \right) C \lg \lambda \right]^{1/p} \\
&\quad \cdot \max_{n \leqslant j \leqslant [\lambda n]} \left(j^{1/q} |\Delta \hat{f}(j)|^{1/q} \right) \max_{n \leqslant j \leqslant [\lambda n]} |\hat{f}(j)|^{1/p}.
\end{aligned}$$

Therefore from (3.1) after taking limit superior as $n \to \infty$ followed by the limit as $\lambda \to 1 + 0$ the proof of the theorem is obtained.

In the case of the complex montone coefficients we have the corresponding theorem where condition (3.1) is slightly weakened.

THEOREM 3.2. *Let*

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int}$$

be the Fourier series of $f \in L^1(T)$ with asymptotically even coefficients.

If $\{\hat{f}(n)\}$ is complex monotone and if for some $1 < p \leqslant 2$,

$$\lim_{\lambda\to 1+0} \overline{\lim_{n\to\infty}} \max_{n\leqslant j\leqslant[\lambda n]} \left(j|\Delta\hat{f}(j)|\right)^{1/q} |\hat{f}([\lambda n]) - \hat{f}(n)|^{1/p} = 0 \tag{3.2}$$

then

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$\hat{f}(n)\lg|n| = o(1), \qquad n \to \infty.$$

PROOF. Using a similar inequality as in Theorem 3.1 we have

$$\left(\sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p} \leqslant \max_{n\leqslant j\leqslant[\lambda n]} \left(j^{1/q}|\Delta\hat{f}(j)|^{1/q}\right)\left(\frac{|\hat{f}([\lambda n]) - \hat{f}(n)|}{\cos\theta_0}\right)^{1/p}$$

from which the proof of Theorem 3.2 follows.

COROLLARY 3.1. *Let*

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int}$$

be the Fourier series of $f \in L^1(T)$ with even coefficients.

If $\{\hat{f}(n)\}$ is complex quasimonotone and if for some $1 < p \leqslant 2$, (3.1) *holds, then* (1.1) *converges in L^1-norm if and only if*

$$\hat{f}(n)\lg|n| = o(1), \qquad n \to \infty.$$

Since

$$n\Delta\hat{f}(n) = o(1), \qquad n \to \infty,$$

implies (3.1), it is clear that the aforementioned result of Č. V. Stanojević [3] is a special case of Corollary 3.1.

4. Additional results and remarks. The conditions (AE1) and (AE2) can be rewritten as

$$\frac{1}{[n/l_n]} \sum_{j=n}^{n+[n/l_n]} |f(j) - f(-j)|\lg j = o(1), \qquad n \to \infty, \tag{4.1}$$

$$\sum_{j=n}^{n+[n/l_n]} |\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j = o(1), \qquad n \to \infty, \tag{4.2}$$

where $l_n \to +\infty$, $l_n = o(n)$, $n \to \infty$ and

$$\left\|\sigma_{n+[n/l_n]}(f) - \sigma_n(f)\right\| l_n = o(1), \qquad n \to \infty \tag{4.3}$$

($\sigma_n(f)$ denotes the Fejér sums). It is easy to see that (4.3) is implied by

$$\|\sigma_n(f) - f\| l_n^{1/2} = o(1), \qquad n \to \infty. \tag{4.4}$$

W. O. Bray and Č. V. Stanojević [2] used (4.4) to obtain a result relating the smoothness of f with the smoothness of $\{\hat{f}(n)\}$. They considered the series (1.1) with coefficients satisfying (4.1) and (4.2) with $l_n = \|\sigma_n(f) - f\|^{-1}$ and proved that if

$$n\Delta\hat{f}(n)\|\sigma_n(f) - f\| = o(1), \qquad n \to \infty, \tag{4.5}$$

then the series (1.1) converges in L^1-norm if and only if

$$\hat{f}(n)\lg\big(|n|\,\|\sigma_n(f) - f\|\big) = o(1), \qquad n \to \infty \tag{4.6}$$

(clearly, the only interesting case is $n\|\sigma_n(f) - f\| \to \infty$, $n \to \infty$).

In the next theorem we shall improve (4.5) assuming that $\{\hat{f}(n)\}$ is a complex quasimonotone sequence.

Theorem 4.1. *Let*

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int}$$

be the Fourier series of $f \in L^1(T)$, satisfying (4.1) *and* (4.2) *with* $l_n = \|\sigma_n(f) - f\|^{-1}$.

If for some $1 < p \leqslant 2$, $(\frac{1}{p} + \frac{1}{q} = 1)$,

$$n^{1/q}|\Delta\hat{f}(n)|^{1/q}\|\sigma_n(f) - f\|^{1/q}|\hat{f}(n)|^{1/p} = o(1), \qquad n \to \infty, \tag{4.7}$$

then

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty,$$

if and only if (4.6) *holds.*

Proof. From the basic inequality in [1] we have

$$\|\sigma_n(f) - f\|^{1/q}\left(\sum_{j=n}^{n+[n\|\sigma_n(f)-f\|]} j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p}$$
$$\leqslant An^{1/q}\|\sigma_n(f) - f\|^{1/q} \max_{n\leqslant j\leqslant n+[n\|\sigma_n(f)-f\|]} |\Delta\hat{f}(j)|^{1/q} \max_{n\leqslant j\leqslant n+[n\|\sigma_n(f)-f\|]} |\hat{f}(j)|^{1/p},$$

where A is an absolute constant. Hence, for $[\lambda n] = n + [n\|\sigma_n(f) - f\|]$, (HK) is satisfied and the proof of the theorem follows.

Since (4.7) is implied by

$$n\Delta\hat{f}(n)|f(n)|^{q-1} = O(1), \qquad n \to \infty, \tag{4.8}$$

we have a corollary to Theorem 4.1.

Corollary 4.1. *Let*

$$S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int}$$

be the Fourier series of $f \in L^1(T)$, satisfying (4.1) *and* (4.2) *with* $l_n = \|\sigma_n(f) - f\|^{-1}$.

If for some $q \geqslant 2$, (4.8) *holds, then*

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty,$$

if and only if (4.6) *holds.*

Thus, in the case of complex quasimonotone coefficients, we have results sharper than the corresponding results in [**1** and **2**].

Our final remark concerns the method (HK) used in this paper. It seems that there should be a direct proof of our Theorems 3.1 and Theorem 4.1 independent of Tauberian condition (HK).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-ROLLA, ROLLA, MISSOURI 65401