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NUCLEAR FACES OF STATE SPACES OF C^*-ALGEBRAS

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ABSTRACT. Let B_1 and B_2 be maximal abelian subalgebras of C^*-algebras A_1 and A_2, and suppose that for each pure state ψ_1 of B_1, the von Neumann algebra $p_{\psi_1}A_1^{**}p_{\psi_1}$ is injective, where p_{ψ_1} is the common support in A_1^{**} of all the states of A_1 which extend ψ_1. Then $B_1 \otimes B_2$ is a maximal abelian subalgebra of any C^*-tensor product $A_1 \otimes_\beta A_2$.

1. Introduction. Let A_1 and A_2 be C^*-algebras with maximal abelian C^*-subalgebras (masas) B_1 and B_2 respectively. The unique C^*-tensor product $B = B_1 \otimes B_2$ of B_1 and B_2 may be regarded as a C^*-subalgebra of any C^*-tensor product $A_1 \otimes_\beta A_2$. Wassermann **[18]** showed that B is a masa in the minimal (spatial) tensor product $A_1 \otimes_{\min} A_2$, and asked whether B is a masa in all the tensor products. The corresponding result for von Neumann algebras is an easy consequence of the Commutation Theorem [**16**, p. 229]. The answer to Wassermann's question is trivially affirmative if A_1 (or A_2) is nuclear (equivalently A_1^{**} (or A_2^{**}) is injective). The purpose of this note is to show that the answer remains affirmative even if the assumption of nuclearity is replaced by a much weaker "nuclear extension property" (NEP) of the subalgebra B_1, which requires only the injectivity of certain small parts of A_1^{**}. The (NEP) is also much weaker than the "extension property" (EP) of B_1— the property that pure states of B_1 (including the zero functional) have unique extensions to pure state of A_1 [**1, 2, 4, 5**]. Indeed the (NEP) holds if no pure state of B_1 has two distinct equivalent pure state extensions to A_1, a property which it is natural to call the "simplex extension property" (SEP) because of its connections with the geometry of the state space of A_1 [**6, 7, 8**].

2. Nuclear faces. For a C^*-algebra A, let $Q(A)$ be the quasi-state space of A in the weak* topology:

$$Q(A) = \{\phi \in A^*: \phi \geqslant 0, \|\phi\| \leqslant 1\}.$$

For a closed face F of $Q(A)$, there is a unique closed projection p^F in A^{**} such that

$$F = \{\phi \in Q(A): \phi(1 - p^F) = 0\}$$

[**15**, 3.11.10]. The linear span of F in A^* is the predual of the von Neumann algebra $p^F A^{**} p^F$. For ϕ in F, let $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ be the corresponding cyclic representation of A^{**}, and let p_ϕ^F be the orthogonal projection of $\mathcal{H}_\phi$ onto the linear subspace of all

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vectors η in $\mathcal{H}_\phi$ for which the vector functional $a \to \langle \pi_\phi(a)\eta, \eta\rangle$ lies in the cone generated by F. Then $p_\phi^F = \pi_\phi(p^F)$, and the following conditions have been shown to be equivalent [6, 7]:

(A1) $p^F A^{**} p^F$ is abelian.

(A2) F is a Choquet simplex.

(A3) For each ϕ in F, $p_\phi^F \pi_\phi(A)'' p_\phi^F$ is abelian.

(A4) For each ϕ in F, $\pi_\phi(A)'$ is abelian.

(A5) Distinct pure states in F are inequivalent.

If these conditions are satisfied, F is said to be *abelian*.

Various classes of faces can be defined by replacing abelianess in (A1) by other algebraic properties of the von Neumann algebra $p^F A^{**} p^F$, for example its type (in the sense of Murray and von Neumann). Another possibility is injectivity, and for this it is easy to obtain analogues of conditions (A2)–(A4). An affine mapping of F into F is a *morphism* if it has a completely positive linear extension to the subspace of A^* spanned by F, and it takes states into states.

PROPOSITION 1. *The following conditions on a closed face F of $Q(A)$ are equivalent*:

(N1) *$p^F A^{**} p^F$ is injective.*

(N2) *There is a net of morphisms of F of finite rank converging to the identity on F in the point-norm topology.*

(N3) *For each ϕ in F, $p_\phi^F \pi_\phi(A)'' p_\phi^F$ is injective.*

(N4) *For each ϕ in F, $\pi_\phi(A)'$ is injective.*

(N4)′ *For each ϕ in F, $\pi_\phi(A)''$ is injective.*

PROOF. (N1) $\Leftrightarrow$ (N2). The linear span of F is identified with the predual of $p^F A^{**} p^F$. Condition (N2) is then equivalent to the semidiscreteness of $p^F A^{**} p^F$ [13], and hence to injectivity [11] (see also [12, 19]).

(N1) $\Rightarrow$ (N3). $p_\phi^F \pi_\phi(A)'' p_\phi^F = \pi_\phi(p^F A^{**} p^F)$, which is a direct summand of $p^F A^{**} p^F$.

(N3) $\Rightarrow$ (N1). If $\{\phi_\lambda : \lambda \in \Lambda\}$ is a maximal family of states in F with orthogonal central supports, then $p^F A^{**} p^F \cong \bigoplus p_{\phi_\lambda}^F \pi_{\phi_\lambda}(A)'' p_{\phi_\lambda}^F$, and so $p^F A^{**} p^F$ is injective.

(N3) $\Rightarrow$ (N4). The commutant of an injective von Neumann algebra is injective [17], so $\pi_\phi(A)' p_\phi^F$ is injective. But the central support of p_ϕ^F in $\pi_\phi(A)''$ is the identity, so $\pi_\phi(A)'$ is isomorphic to $\pi_\phi(A)' p_\phi^F$.

(N4) $\Rightarrow$ (N4)′. This is immediate from the result of [17].

(N4)′ $\Rightarrow$ (N3). This is immediate.

DEFINITION 2. A closed face F of $Q(A)$ is *nuclear* if the equivalent conditions (N1)–(N4)′ hold.

The remarkable characterisation of nuclear C^*-algebras obtained by Choi and Effros [9, 10] from the results of Connes [11] can now be phrased as

$$A \text{ is nuclear} \Leftrightarrow Q(A) \text{ is nuclear}.$$

3. The NEP and tensor products. It is now possible to study nuclearity of the face associated with a single pure state of a C^*-subalgebra B of A. Thus if ψ is an extreme point of $Q(B)$ (either a pure state or the zero functional), let

$$Q_\psi(A) = \{\phi \in Q(A) : \phi|_B = \psi\}.$$

Then $Q_\psi(A)$ is a nonempty closed face of $Q(A)$. The C^*-subalgebra B is said to have the *extension property* (EP) in A if $Q_\psi(A)$ contains only one functional, for each extreme point ψ of $Q(B)$ [**1, 4, 5**].

DEFINITION 3. A C^*-subalgebra B of A has the *simplex* (respectively, *nuclear*) *extension property* (SEP (respectively, NEP)) in A, if for each extreme point ψ of $Q(B)$, the face $Q_\psi(A)$ of $Q(A)$ is abelian (respectively, nuclear).

Clearly (EP) $\Rightarrow$ (SEP) $\Rightarrow$ (NEP). Abelian C^*-subalgebras with the (EP) have been extensively studied in [**1, 4, 5**]; nonatomic masas in type I factors do not have the (SEP) [**14, 1**], but it does not seem to be easy to find masas which do not have the (NEP).

The connection between the (NEP) and tensor products is established in the following lemma.

LEMMA 4. *Let $A_1 \otimes_\beta A_2$ be a C^*-tensor product of C^*-algebras A_1 and A_2, let Ψ: $A_1 \otimes_\beta A_2 \to A_1 \otimes_{\min} A_2$ be the canonical *-homomorphism, and let ϕ be a state of $A_1 \otimes_\beta A_2$ whose restriction to A_1 lies in a nuclear face of $Q(A_1)$. Then there is a state $\phi_{\min}$ of $A_1 \otimes_{\min} A_2$ such that $\phi = \phi_{\min} \circ \Psi$.*

PROOF. There is a faithful embedding Φ of the algebraic tensor product $A_1^{**} \odot A_2^{**}$ in $(A_1 \otimes_\beta A_2)^{**}$ which is normal in each variable separately, and acts as the identity on $A_1 \odot A_2$ (see the proof of [3, Theorem 2]). Let $\psi = \phi \circ \Phi$. The restriction of ϕ to A_1 is given by $a_1 \to \psi(a_1 \otimes \mathbf{1})$, so that there is a (closed) projection p in A_1^{**} with $\psi(p \otimes \mathbf{1}) = 1$ such that $pA_1^{**}p$ is injective, hence semidiscrete [**11**]. Now ψ is a normalised positive linear functional on $A_1^{**} \odot A_2^{**}$ which is normal in each variable separately, so ψ is continuous and of norm 1 for the minimal C^*-norm on $pA_1^{**}p \odot A_2$ [**13**, Theorem 4.1]. Hence

$$|\phi(x)| = |\psi((p \otimes \mathbf{1})x(p \otimes \mathbf{1}))| \leq \|(p \otimes \mathbf{1})x(p \otimes \mathbf{1})\|_{\min} \leq \|x\|_{\min} \qquad (x \in A_1 \odot A_2).$$

Thus ϕ factors through $A_1 \otimes_{\min} A_2$.

Minor modifications of the proof of Lemma 4 show that if the restriction of ϕ to A_1 lies in an abelian face of $Q(A_1)$, then ϕ factors through the tensor product $A_1 \otimes_\lambda A_2$ of A_1 and A_2 in the least cross-norm of Banach spaces. Hence if B_1 is an abelian C^*-subalgebra with the (SEP) in A_1, and B_2 is a C^*-subalgebra with the (SEP) in A_2, then $B_1 \otimes B_2$ has the (SEP) in $A_1 \otimes_\beta A_2$.

It is very easy to see that if B_1 is an abelian C^*-subalgebra with the (EP) in A_1 and B_2 is any C^*-subalgebra with the (EP), (SEP) or (NEP) in A_2, then $B_1 \otimes B_2$ has the (EP), (SEP) or (NEP) respectively in $A_1 \otimes_\beta A_2$. Hence if B_1 and B_2 are masas with the (EP), then $B_1 \otimes B_2$ is a masa. The following result generalises this.

THEOREM 5. *Let $A = A_1 \otimes_\beta A_2$ be any C^*-tensor product of C^*-algebras A_1 and A_2, let B_j be a masa of A_j ($j = 1, 2$), and $B = B_1 \otimes B_2$. Suppose that B_1 has the (NEP) in A_1. Then B is a masa in A.*

PROOF. Let C be an abelian C^*-subalgebra of A containing B, and Ψ: $A \to A_1 \otimes_{\min} A_2$ be the canonical *-homomorphism, so that Ψ is the identity on B. Then

$\Psi(C)$ is an abelian C^*-subalgebra of $A_1 \otimes_{\min} A_2$ containing B, so $\Psi(C) = B$ **[18**, Corollary 6].

Let ϕ and ϕ' be multiplicative linear functionals on C which coincide on B, and ψ and ψ' be norm-preserving extensions of ϕ and ϕ' to functionals in $Q(A)$. Since the restrictions of ϕ and ϕ' to B_1 are multiplicative, it follows from Lemma 4 that there are functionals $\phi_{\min}$ and $\phi'_{\min}$ in $Q(A_1 \otimes_{\min} A_2)$ such that $\phi = \phi_{\min} \circ \Psi$, $\phi' = \phi'_{\min} \circ \Psi$. Then $\phi_{\min}$ and $\phi'_{\min}$ coincide on $B = \Psi(C)$, so ϕ and ϕ' coincide on C, and $\psi = \psi'$. It follows from the Stone-Weierstrass Theorem that $B = C$, so B is a masa in A.

The question whether Theorem 5 is valid without the assumption of the (NEP) remains open.

REFERENCES

1. J. Anderson, *Extensions, restrictions, and representations of states in C^*-algebras*, Trans. Amer. Math. Soc. **249** (1979), 303–329.

2. ______, *Extreme points in sets of positive linear maps on $\mathfrak{B}(\mathfrak{H})$*, J. Funct. Anal. **31** (1979), 195–217.

3. R. J. Archbold, *On the centre of a tensor product of C^*-algebras*, J. London Math. Soc. (2) **10** (1975), 257–262.

4. ______, *Extensions of states of C^*-algebras*, J. London Math. Soc. (2) **21** (1980), 351–354.

5. R. J. Archbold, J. W. Bunce and K. Gregson, *Extensions of states of C^*-algebras*. II, Proc. Roy. Soc. Edinburgh (to appear).

6. C. J. K. Batty, *Simplexes of states of C^*-algebras*, J. Operator Theory **4** (1980), 3–23.

7. ______, *Abelian faces of state spaces of C^*-algebras*, Comm. Math. Phys. **75** (1980), 43–50.

8. ______, *Simplexes of extensions of states of C^*-algebras*, Trans. Amer. Math. Soc. **272** (1982), 237–246.

9. M. D. Choi and E. G. Effros, *Separable nuclear C^*-algebras and injectivity*, Duke Math. J. **43** (1976), 309–322.

10. ______, *Nuclear C^*-algebras and injectivity: the general case*, Indiana Univ. Math. J. **26** (1977), 443–446.

11. A. Connes, *Classification of injective factors*, Ann. of Math. (2) **109** (1976), 73–115.

12. ______, *On the equivalence between injectivity and semi discreteness for operator algebras*, Colloques Internat. CNRS **274** (1979), 107–112.

13. E. G. Effros and E. C. Lance, *Tensor products of operator algebras*, Adv. in Math. **25** (1977), 1–34.

14. R. V. Kadison and I. M. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400.

15. G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London and New York, 1979.

16. M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, Berlin and New York, 1979.

17. J. Tomiyama, *Tensor products and projections of norm one in von Neumann algebras*, Seminar notes, Copenhagen Univ., 1971.

18. A. S. Wassermann, *The slice map problem for C^*-algebras*, Proc. London Math. Soc. (3) **32** (1976), 537–559.

19. ______, *Injective W^*-algebras*, Math. Proc. Cambridge Philos. Soc. **82** (1977), 39–47.

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