

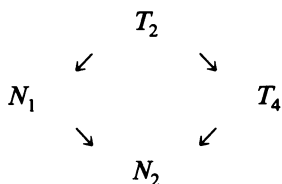
# SPLITTING $S^4$ ON $\mathbf{R}P^2$ VIA THE BRANCHED COVER OF $\mathbf{C}P^2$ OVER $S^4$

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**ABSTRACT.** The four sphere decomposes as a twisted double  $N_2 \cup_f N_2$ , where  $N_2$  is the 2-disk bundle over the real projective plane with Euler number 2. In this note the relationship of this splitting to the double branched cover of the complex projective plane over the four sphere as the quotient space under complex conjugation is made explicit.

It is known that the quotient of  $\mathbf{C}P^2$  under complex conjugation is  $S^4$  (cf. [1, 2, 4]). While trying to understand this the author was led to the splitting  $S^4 = N_2 \cup_f N_2$ , where  $N_2$  is a tubular neighborhood of a standard  $\mathbf{R}P^2$  in  $S^4$  [3] (see [5, 6] for other approaches to this splitting).  $N_2$  can be characterized as being the nonorientable disk bundle over  $\mathbf{R}P^2$  with Euler number 2. In this note, we would like to make explicit the relationship of the splitting of  $S^4$  to the branched cover of  $\mathbf{C}P^2$  over  $S^4$ . This proof also exhibits an interesting decomposition of  $\mathbf{C}P^2$ .

Let  $T_k$  denote the 2-disk bundle over  $S^2$  with Euler number  $k$ ;  $T_2$  is just the tangent bundle of  $S^2$ . Let  $N_k$  be the nonorientable 2-disk bundle over  $\mathbf{R}P^2$  with Euler number  $k$ ;  $N_1$  is the tangent bundle of  $\mathbf{R}P^2$ . Here the descriptions of  $T_4$ ,  $N_1$ ,  $N_2$  based on the usual description of  $T_2 = \{(x, y) \in S^2 \times D^3: x \cdot y = 0\}$ . Then  $T_4 = T_2/(x, y) \sim (x, -y)$ ,  $N_1 = T_2/(x, y) \sim (-x, y)$ , and  $N_2 = T_2/(x, y) \sim (\pm x, \pm y)$ . Note that the quotient maps lead to a commutative diagram:



Our starting point is Massey's approach to  $\mathbf{C}P^2/\sim \approx S^4$  [4]. Massey regards  $\mathbf{C}P^2 = S^2 \times S^2/(x, y) \sim (y, x)$ , with complex conjugation being equivalent to  $[(x, y)] \rightarrow [(-x, -y)]$ . He then shows that  $S^2 \times S^2/H \approx S^4$ , where  $H = \{I, (x, y) \rightarrow (y, x), (x, y) \rightarrow (-x, -y), (x, y) \rightarrow (-y, -x)\}$ . For completeness, we will give the details of  $S^2 \times S^2/(x, y) \sim (y, x) \approx \mathbf{C}P^2$  since they are not given in [4]. One regards  $S^2$  as  $\mathbf{C} \cup \{\infty\}$  via stereographic projection. Then for a pair of complex

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numbers  $z_1, z_2$  one thinks of  $z_1, z_2$  as the roots of the polynomial  $(z - z_1)(z - z_2) = z_1 z_2 - (z_1 + z_2)z + z^2$  and sends  $\{z_1, z_2\}$  to the coefficients of this polynomial, regarded as an element of  $\mathbf{C}P^2$ ,  $\{z_1, z_2\} \rightarrow [z_1 z_2, -(z_1 + z_2), 1]$ . The pair  $\{z_1, \infty\}$  is sent to the coefficients of  $z - z_1$  (i.e. to  $[-z_1, 1, 0]$ ) and  $\{\infty, \infty\}$  goes to the coefficients of 1 (i.e. to  $[1, 0, 0]$ ). One checks that this gives a diffeomorphism  $S^2 \times S^2/(x, y) \sim (y, x) \rightarrow \mathbf{C}P^2$ . One then computes that the involution  $(x, y) \rightarrow (-x, -y)$  of  $S^2 \times S^2$  induces the involution  $t'[z_1, z_2, z_3] = [\bar{z}_3, -\bar{z}_2, \bar{z}_1]$  of  $\mathbf{C}P^2$ . This is equivalent to complex conjugation  $t[z_1, z_2, z_3] = [\bar{z}_1, \bar{z}_2, \bar{z}_3]$ . An equivalence is given by  $h: \mathbf{C}P^2 \rightarrow \mathbf{C}P^2$ ,  $h[z_1, z_2, z_3] = (1/\sqrt{2})(iz_1 + z_3), \zeta z_2, 1/\sqrt{2}(z_1 + iz_3))$ ,  $\zeta^2 = -i$ ; one computes that  $hth^{-1} = t'$ . Using this equivalence, we will regard  $\mathbf{C}P^2$  as  $S^2 \times S^2/(x, y) \sim (y, x)$  from now on and conjugation as being given by  $[(x, y)] \rightarrow [(-x, -y)]$ .

We first decompose  $S^2 \times S^2$  as  $A \cup B$ , where  $A = \{(x, y): x \cdot y \geq 0\}$  and  $B = \{(x, y): x \cdot y \leq 0\}$ . If  $\alpha(x, y) = (x, -y)$ , then  $\alpha$  is a diffeomorphism of  $S^2 \times S^2$  interchanging  $A$  and  $B$ . Note that  $A$  is diffeomorphic to  $T_2$ , being a tubular neighborhood of the diagonal. An explicit diffeomorphism is given by  $g: T_2 \rightarrow A$ ,  $g(x, y) = (\exp_x(y/2), \exp_x(-y/2))$ , where  $\exp_x v$  goes along the great circle from  $x$  in direction  $v$  a distance  $(\pi/2)|v|$ . This leads to a decomposition  $S^2 \times S^2 = A \cup B \approx T_2 \cup_{\alpha'} T_2$ , where  $\alpha' = g^{-1}\alpha g$ ,  $\alpha'(x, y) = (y, x)$ . Here  $T_2$  is mapped to  $A$  via  $g$  on the left and is mapped to  $B$  via  $\alpha g$  on the right. Our convention is that glueing maps go from right to left.

Now  $\beta(x, y) = (y, x)$  on  $S^2 \times S^2$  becomes  $\beta' = g^{-1}\beta g$ ,  $\beta'(x, y) = (x, -y)$  on the left copy of  $T_2$  and  $\beta'' = g^{-1}\alpha^{-1}\beta\alpha g$ ,  $\beta''(x, y) = (-x, y)$  on the right copy of  $T_2$ . Thus  $S^2 \times S^2/\beta = \mathbf{C}P^2$  decomposes as  $T_2/\beta' \cup_{\bar{\alpha}} T_2/\beta''$ , where  $\bar{\alpha}[(x, y)] = [(y, x)]$ . But  $T_2/\beta' = T_4$  and  $T_2/\beta'' = N_1$ . Thus  $\mathbf{C}P^2$  decomposes as  $T_4 \cup_{\bar{\alpha}} N_1$ .

We have seen that complex conjugation on  $\mathbf{C}P^2$  is equivalent to the involution induced by  $\gamma(x, y) = (-x, -y)$  on  $S^2 \times S^2$ , which corresponds to  $\gamma', \gamma''$  on the two factors of  $T_2$ ,  $\gamma'(x, y) = (-x, -y) = \gamma''(x, y)$ . Thus its quotient, which is  $S^4$ , splits as

$$T_2/\{\beta', \gamma'\} \cup T_2/\{\beta'', \gamma''\} = N_2 \cup_f N_2,$$

with  $f[(x, y)] = [(y, x)]$ .

The following diagram summarizes these splittings:

$$\begin{array}{ccc} S^2 \times S^2 & \approx & T_2 \cup_{\alpha'} T_2 \\ \downarrow / \beta & & \\ \mathbf{C}P^2 & \approx & T_4 \cup_{\bar{\alpha}} N_1 \\ \downarrow / \gamma & & \\ S^4 & \approx & N_2 \cup_f N_2 \end{array}$$

REMARKS. (1) Note that [1, 2, 4] show  $\mathbf{C}P^2/\sim \approx S^4$  while [3, 5, 6] show  $N \cup_f N \approx S^4$ . Another simple proof that  $\mathbf{C}P^2/\sim \approx S^4$  can be based on examining the standard handle decomposition  $\mathbf{C}P^2 \approx h^0 \cup h^2 \cup h^4$ . Under complex conjugation, the quotient gives the union of 3 disks. The first two are glued together along a common

boundary disk to give  $D^4$  and the last one is glued along the full boundary  $S^3$  to this to give  $S^4$ .

(2) In the decomposition  $CP^2 \approx T_4 \cup_{\bar{\alpha}} N_1$ , the  $RP^2 \subset N_1$  is just the standard  $RP^2 \subset CP^2$ . One can check, using the equivalence  $h$  above, that  $S^2 \subset T_4$  is given by the solution of  $2z_1^2 + z_2^2 + 2z_3^2 = 0$ . This represents  $2[CP^1]$  homologically.

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