PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 4, December 1982

A GENERALIZATION OF A THEOREM OF AYOUB AND CHOWLA

DON REDMOND

ABSTRACT. Let χ_1 and χ_2 be characters modulo q_1 and q_2, respectively, where q_1 and q_2 are positive integers. Let

$$f(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).$$

In this paper we shall give an estimate for the sum

$$\sum_{n\leq x} f(n)\log(x/n).$$

1. Introduction. Let χ_1 and χ_2 be characters modulo q_1 and q_2, respectively, where q_1 and q_2 are positive integers. Let

$$f(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).$$

In this paper we shall give an estimate for the sum

$$\sum_{n\leq x} f(n)\log(x/n).$$

In [2] Ayoub and Chowla considered the case $\chi_1 \equiv 1$, $q_1 = 1$ and χ_2 the Kronecker character. In [3] Müller considered the case $\chi_1 \equiv 1$ and χ_2 the nonprincipal character modulo 4.

We shall prove the following theorem.

THEOREM. *We have, as $x \to \infty$,*

$$\sum_{n\leq x} f(n)\log(x/n) = C_1(\chi_1,\chi_2)x\log x + C_2(\chi_1,\chi_2)x + C_3(\chi_1,\chi_2)\log x + C_4(\chi_1,\chi_2) + O(x^{-2/3}),$$

where the $C_j(\chi_1,\chi_2)$, $1 \leq j \leq 4$, are certain constants to be determined below (see (3.10) and (4.5)).

In what follows we shall use the notation $\int_{(a,T)}$ to stand for the integral $\int_{a-iT}^{a+iT}$.

2. Lemmas.

LEMMA 1. *Let χ be a character modulo q and let $L(s,\chi)$ be the associated Dirichlet L series. Then, for $s \neq 1$, we have*

$$L(s,\chi) = q^{-s}\sum_{n=1}^{q} \chi(n)\varsigma(s,n/q),$$

where $\varsigma(s,n/q)$ is the Hurwitz zeta function.

Received by the editors June 3, 1977.
1980 *Mathematics Subject Classification*. Primary 10H25; Secondary 10H10.
Key words and phrases. Character sums, L-functions.

PROOF. For $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty} \chi(n)n^{-s} = \sum_{m=0}^{\infty}\sum_{n=1}^{q} \chi(mq+n)(mq+n)^{-s} \\ &= \sum_{n=1}^{q} \chi(n) \sum_{m=0}^{\infty} (mq+n)^{-s} = q^{-s} \sum_{n=1}^{q} \chi(n)\varsigma(s,n/q). \end{aligned}$$

Now $\varsigma(s,n/q)$ can be continued to a meromorphic function in the entire complex plane whose only singularity is a simple pole at $s=1$ (see [**5**, §13.13]). Since $L(s,\chi)$ can be continued to a meromorphic function in the complex plane with at most a simple pole at $s=1$ the result follows by analytic continuation and completes the proof of the lemma.

LEMMA 2. (1) *Let* $0 < a \le 1$. *Then we have, as* $|t| \to \infty$,

$$\varsigma(\sigma+it,a) \ll \begin{cases} 1 & \text{if } \sigma > 1, \\ |t|^{(1-\sigma)/2}\log|t| & \text{if } 0 \le \sigma \le 1, \\ |t|^{1/2-\sigma}\log|t| & \text{if } \sigma < 0. \end{cases}$$

(2) *Let* $L(s,\chi)$ *be as in Lemma* 1. *Then we have, as* $|t| \to \infty$,

$$L(\sigma+it,\chi) \ll \begin{cases} 1 & \text{if } \sigma > 1, \\ (q|t|)^{(1-\sigma)/2}\log(q|t|) & \text{if } 0 \le \sigma \le 1, \\ (q|t|)^{1/2-\sigma}\log(q|t|) & \text{if } \sigma < 0. \end{cases}$$

PROOF. (1) is proved in [**5**, p. 276] and (2) follows from (1) easily by use of Lemma 1.

LEMMA 3. *Let* $L(s,\chi)$ *be as in Lemma* 1. *Let*

$$G(n,\chi) = \sum_{m=1}^{q} \chi(m)\exp(2\pi i mn/q).$$

For $\mathrm{Re}(s) > 1$, *let*

$$L^*(s,\chi) = \sum_{n=1}^{\infty} G(n,\chi)n^{-s}.$$

Then $L(s,\chi)$ *and* $L^*(s,\chi)$ *satisfy the functional equation*

$$L(s,\chi) = (i/q)(2\pi/q)^{s-1}\Gamma(1-s)\left\{e^{\pi i s/2} + \chi(-1)e^{-\pi i s/2}\right\}L^*(1-s,\chi).$$

This result can be found in [**1**, Theorems 1 and 3].

3. Proof of the theorem. For $\mathrm{Re}(s) > 1$, if we let

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}, \tag{3.1}$$

then we have

$$F(s) = L(s,\chi_1)L(s,\chi_2). \tag{3.2}$$

By Perron's formula, we have, by (3.1),

$$\sum_{n\le x} f(n)\log(x/n) = \frac{1}{2\pi i}\int_{(1+1/\log x, T)} F(s)x^s s^{-2}\,ds + O(x^\delta/T + x^{1+\delta}\log x/T^2), \tag{3.3}$$

for any fixed $\delta > 0$.

Let $T > 1$ and R_T be the rectangle with vertices: $1+1/\log x \pm iT$ and $-1/14 \pm iT$. Then on the interior of $R_T F(s)x^s s^{-2}$ has a double pole at $s = 0$ and a double, single or no pole at $s = 1$, depending on whether both, one or none of the characters, χ_1 and χ_2, are principal characters. On $R_T x^s F(s)s^{-2}$ is analytic. Thus, by the residue theorem, we have

$$\frac{1}{2\pi i}\int_{R_T} F(s)x^s s^{-2}\,ds = \operatorname{res}(F(s)s^{-2}x^s, 0) + \operatorname{res}(F(s)s^{-2}x^s, 1). \tag{3.4}$$

In a neighborhood of $s = 0$ we have

$$\begin{aligned} x^s s^{-2} f(s) &= s^{-2}(1 + s\log x + \tfrac{1}{2}s^2 + O(s^3))(F(0) + F'(0)s + O(s^2)) \\ &= F(0)s^{-2} + (F(0)\log x + F'(0))s^{-1} + O(1). \end{aligned}$$

Thus

$$\operatorname{res}(x^s s^{-2}F(s), 0) = F(0)\log x + F'(0). \tag{3.5}$$

If neither character is principal, then

$$\operatorname{res}(x^s F(s)s^{-2}, 1) = 0. \tag{3.6}$$

Suppose χ_1 is a principal character modulo q_1 and χ_2 is a nonprincipal character modulo q_2. Then, in a neighborhood of $s = 1$, we have

$$\begin{aligned} L(s,\chi_1) &= \gamma_{-1}(q_1)(s-1)^{-1} + \gamma(q_1) + O(s-1), \\ L(s,\chi_2) &= L(1,\chi_2) + L'(1,\chi_2)(s-1) + O(s-1)^2, \\ x^s &= x(1 + (s-1)\log x + O(s-1)^2) \end{aligned}$$

and

$$s^2 = 1 - (s-1) + O(s-1)^2.$$

Combining these expansions gives

$$\operatorname{res}(x^s F(s)s^{-2}, 1) = L(1,\chi_2)\gamma_{-1}(q_1)x. \tag{3.7}$$

Similarly, if χ_1 is a nonprincipal character modulo q_1 and χ_2 is the principal character modulo q_2, then

$$\operatorname{res}(x^s F(s)s^{-2}, 1) = L(1,\chi_1)\gamma_{-1}(q_2)x. \tag{3.8}$$

If both χ_1 and χ_2 are principal characters, then we have, as above,

$$\begin{aligned} \operatorname{res}(x^s F(s)s^{-2}, 1) &= \gamma_{-1}(q_1)\gamma_{-1}(q_2)x\log x \\ &\quad + (\gamma_{-1}(q_1)\gamma(q_2) + \gamma_{-1}(q_2)\gamma(q_1) - \gamma_{-1}(q_1)\gamma_{-1}(q_2))x. \end{aligned} \tag{3.9}$$

Let

$$(3.10)\qquad \begin{aligned} C_1(\chi_1,\chi_2) &= \begin{cases} \gamma_{-1}(q_1)\gamma_{-1}(q_2) & \text{if } \chi_1 \text{ and } \chi_2 \text{ are principal,} \\ 0 & \text{else,} \end{cases} \\ C_2(\chi_1,\chi_2) &= \begin{cases} -\gamma_{-1}(q_1)\gamma_{-1}(q_2)+\gamma_{-1}(q_1)\gamma(q_2)+\gamma_{-1}(q_2)\gamma(q_1) \\ \qquad\qquad \text{if both } \chi_1 \text{ and } \chi_2 \text{ are principal,} \\ L(1,\chi_2)\gamma_{-1}(q_1) & \text{if } \chi_1 \text{ principal, } \chi_2 \text{ not,} \\ L(1,\chi_1)\gamma_{-1}(q_2) & \text{if } \chi_2 \text{ principal, } \chi_1 \text{ not,} \\ 0 & \text{neither character principal,} \end{cases} \\ C_3(\chi_1,\chi_2) &= F(0) \quad \text{and} \quad C_4(\chi_1,\chi_2)=F'(0). \end{aligned}$$

Then, by (3.4)–(3.10), we have

$$(3.11)\qquad \frac{1}{2\pi i}\int_{R_T} F(s)x^s s^{-2}\,ds = C_1(\chi_1,\chi_2)x\log x + C_2(\chi_1,\chi_2)x + C_3(\chi_1,\chi_2)\log x + c_4(\chi_1,\chi_2).$$

By Lemma 2 and (3.2), we have

$$(3.12)\qquad \begin{aligned} &\left|\int_{1+1/\log x\pm iT}^{-1/14\pm iT} x^s F(s)s^{-2}\,ds\right| \\ &= \left|\int_{1+1/\log x}^{-1/14} x^{\sigma\pm iT}(\sigma\pm iT)^{-2}F(\sigma\pm iT)\,d\sigma\right| \\ &= \left|\left\{\int_{1+1/\log x}^{1}+\int_1^0+\int_0^{-1/14}\right\}x^{\sigma\pm iT}(\sigma\pm iT)^{-2}F(\sigma\pm iT)\,d\sigma\right| \\ &\ll x/T^2 + x\log^2 T/T + T^{-6/7}\log^2 T. \end{aligned}$$

To estimate the fourth side of R_T, the integral

$$(3.13)\qquad \int_{(-1/14,T)} x^s F(s)s^{-2}\,ds = \int_{-T}^{T} x^{-1/14+it}(-1/14+it)^2 F(-1/14+it)\,dt,$$

we use the relation (3.2) and the functional equation given in Lemma 3 to rewrite the integral on the right-hand side of (3.13). This is then estimated, by Sterling's formula and a lemma of van der Corput [4, Lemma 4.5], as in [4, pp. 265–266]. The result is

$$(3.14)\qquad \int_{(-1/14,T)} x^s F(s)s^{-2}\,ds \ll T^{-6/7} + x^{-1/14}T^{-5/14}.$$

Thus, (3.3), (3.11), (3.12) and (3.14), we have

$$\begin{aligned} \sum_{n\le x} f(n)\log(x/n) &= C_1(\chi_1,\chi_2)x\log x + C_2(\chi_1,\chi_2)x \\ &\quad + C_3(\chi_1,\chi_2)\log x + C_4(\chi_1,\chi_2) \\ &\quad + O(x^\delta/T + T^{-2}x^{1+6}\log x + x/T^2 + T^{-6/7}\log^2 T \\ &\qquad + x\log^2 T/T + x^{-1/14}T^{-5/14} + T^{-6/7}). \end{aligned}$$

If we choose $T = x^{5/3}$ we get the result of the theorem.
This completes the proof of the theorem.

4. Further computation of the coefficients $C_j(\chi_1, \chi_2)$.

LEMMA 4. *Let χ_0 be the principal character modulo q. If, in a neighborhood of $s=1$,*

$$L(s,\chi_0) = \gamma_{-1}(q)(s-1)^{-1} + \gamma(q) + O_q(s-1),$$

then

$$\gamma_{-1}(q) = \varphi(q)/q \quad \text{and} \quad \gamma(q) = \frac{\varphi(q)}{q}\left\{\sum_{p|q}\frac{\log p}{p-1} + \gamma\right\}, \tag{4.1}$$

where $\varphi(q)$ is Euler's function and γ is Euler's constant.

PROOF. We have

$$L(s,\chi_0) = \prod_{p|q}(1-p^{-s})\zeta(s). \tag{4.2}$$

In a neighborhood of $s=1$, we have

$$\zeta(s) = (s-1)^{-1} + \gamma + O(s-1) \tag{4.3}$$

and

$$\prod_{p|q}(1-p^{-s}) = \frac{\varphi(q)}{q}\left\{1 + \sum_{p|q}\frac{\log p}{p-1}(s-1) + O_q(s-1)\right\}. \tag{4.4}$$

If we combine the representations (4.2)–(4.4), the result, (4.1), follows by comparing coefficients and completes the proof of the lemma.

LEMMA 5. *Let $L(s,\chi)$ be as in Lemma 1. Then*

$$L(0,\chi) = \sum_{n=1}^{q}\chi(n)(\tfrac{1}{2} - n/q)$$

and

$$L'(0,\chi) = \log q\sum_{n=1}^{q}\chi(n)(\tfrac{1}{2} - n/q) + \sum_{n=1}^{q}(\log\Gamma(n/q) - \tfrac{1}{2}\log 2\pi)\chi(n).$$

PROOF. By Lemma 1, we have

$$L(s,\chi) = q^{-s}\sum_{n=1}^{q}\chi(n)\zeta(s,n/q).$$

Thus

$$L'(s,\chi) = q^{-s}\log q\sum_{n=1}^{q}\chi(n)\zeta(s,n/q).$$

Thus

$$L(0,\chi) = \sum_{n=1}^{q}\chi(n)\zeta(0,n/q)$$

and

$$L'(0,\chi) = \log q\sum_{n=1}^{q}\chi(n)\zeta(0,n/q) + \sum_{n=1}^{q}\chi(n)\zeta'(0,n/q).$$

By [5, p. 271], we have

$$\zeta(0,n/q) = \tfrac{1}{2} - n/q \quad \text{and} \quad \zeta'(0,n/q) = \log\Gamma(n/q) - \tfrac{1}{2}\log 2\pi.$$

Combining these results gives the lemma and completes the proof.

Thus, by (3.2), (3.10) and Lemmas 4 and 5, we have

$$C_1(\chi_1,\chi_2)=\begin{cases}\varphi(q_1)\varphi(q_2)/q_1q_2 & \text{if both characters are principal,}\\ 0 & \text{else,}\end{cases}$$

$$C_2(\chi_1,\chi_2)=\begin{cases}\dfrac{\varphi(q_1)\varphi(q_2)}{q_1q_2}\left\{\left(\sum_{p|q_1}+\sum_{p|q_2}\right)\dfrac{\log p}{p-1}+2\gamma-1\right\} & \\ \qquad\qquad\text{if both characters are principal,} & \\ L(1,\chi_2)\varphi(q_1)/q_1 & \text{if } \chi_1 \text{ is principal, } \chi_2 \text{ not,}\\ L(1,\chi_1)\varphi(q_2)/q_2 & \text{if } \chi_2 \text{ is principal, } \chi_1 \text{ not,}\\ 0 & \text{else,}\end{cases}$$

(4.5)

$$C_3(\chi_1,\chi_2)=\sum_{n=1}^{q_1}\chi_1(n)(\tfrac{1}{2}-n/q_1)\sum_{n=1}^{q_2}\chi_2(n)(\tfrac{1}{2}-n/q_2), \text{ and}$$

$$\begin{aligned}C_4(\chi_1,\chi_2)&=\sum_{n=1}^{q_1}\chi_1(n)(\tfrac{1}{2}-n/q_1)\sum_{n=1}^{q_2}\chi_2(n)(\log\Gamma(n/q_2)-\tfrac{1}{2}\log 2\pi)\\&+\sum_{n=1}^{q_1}\chi_1(n)(\log\Gamma(n/q_1)-\tfrac{1}{2}\log 2\pi)\sum_{n=1}^{q_2}\chi_2(n)(\tfrac{1}{2}-n/q_2)\\&+(\log q_1q_2)C_3(\chi_1,\chi_2).\end{aligned}$$

5. Examples.

EXAMPLE 1. Let $\chi_1=\chi_2\equiv 1$ and $q_1=q_2=1$. Then $f(n)=d(n)$, the divisor function. The theorem gives

$$\sum_{n\le x} f(n)\log(x/n)=x\log x+(2\gamma-1)x+\tfrac{1}{4}\log x+\tfrac{1}{2}\log 2\pi+O(x^{-2/3}),$$

as $x\to\infty$.

EXAMPLE 2. Let $\chi_1\neq 1$, $q_1=1$ and $q_2=4$ with χ_2 the nonprincipal character modulo 4. Then $f(n)=\frac{1}{4}r(n)$, where $r(n)$ is the number of representations of n as a sum of two squares. The theorem gives, as $x\to\infty$,

$$\sum_{n\le x} r(n)\log(x/n)=\pi x+\log x-\log(\pi/2)-2\log(\Gamma(\tfrac{1}{4})/\Gamma(\tfrac{3}{4}))+O(x^{-2/3}).$$

This betters the result of Müller [**3**] who obtained an error term of $O(x^{-1/4})$.

EXAMPLE 3. Let $q_1=1$, $\chi_1\equiv 1$ and χ_2 be the Kronecker character modulo $-q$. Then $f(n)$ is the number of integral ideals of norm n in the imaginary quadratic field $Q(\sqrt{-q})$. The theorem gives, as $x\to\infty$,

$$\begin{aligned}\sum_{n\le x} f(n)\log(x/n)&=L(1,\chi_2)x+q^{-1}\sum_{n=1}^{q} n(q/n)\log x\\&-(\tfrac{1}{2}\log 2\pi)q^{-1}\sum_{n=1}^{q} n(q/n)+(\log q/2q)\sum_{n=1}^{q} n(q/n)\\&-\frac{1}{2}\sum_{n=1}^{q}(q/n)\log\Gamma(n/q)+O(x^{-2/3}),\end{aligned}$$

where (q/n) is the Kronecker character. This betters the result of Ayoub and Chowla [**2**] who obtained an error term of $O(x^{-1/4})$.

REFERENCES

1. T. M. Apostol, *Dirichlet L-series and character power sums*, J. Number Theory **2** (1970), 223–234.
2. R. Ayoub and S. Chowla, *On a theorem of Müller and Carlitz*, J. Number Theory **2** (1970), 342–344.
3. C. Müller, *Eine Formel der analytische Zahlentheorie*, Abh. Hamburg Univ. **19** (1954), 62–65.
4. E. C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford Univ. Press, London, 1951.
5. E. T. Whittaker and G. N. Watson, *A course in modern analysis*, 4th ed., Cambridge Univ. Press, London, 1965.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801