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# COHOMOLOGY OF HEISENBERG LIE ALGEBRAS

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ABSTRACT. The cohomology of Heisenberg Lie algebras is studied and we obtain the description of cocycles, coboundaries and cohomological spaces.

## 1. Notations and preliminaries.

1.1. Let $\underline{g}$ be a Lie algebra of dimension $n$ over a field $F$ and $M$ a $\underline{g}$-module of finite dimension over $F$. We denote by $C^p(\underline{g}, M)$ the space of cochains of degree $p$, $d_p: C^p(\underline{g}, M) \to C^{p+1}(\underline{g}, M)$ the restriction to $C^p(\underline{g}, M)$ of the coboundary operator, $Z^p(\underline{g}, M)$ the kernel of $d_p$ (space of cocycles of degree $p$), $B^p(\underline{g}, M)$ the range of $d_{p-1}$ (space of coboundaries of degree $p$), $H^p(\underline{g}, M)$ the quotient of $Z^p(\underline{g}, M)$ by $B^p(\underline{g}, M)$ (space of cohomology of degree $p$ of $\underline{g}$ with values in $M$). If $M = F$, denote $C^p(\underline{g}) = C^p(\underline{g}, F)$, $Z^p(\underline{g}) = Z^p(\underline{g}, F)$, $B^p(\underline{g}) = B^p(\underline{g}, F)$, $H^p(\underline{g}) = H^p(\underline{g}, F)$. For all details see [**1, 2, 5, 6**].

1.2. By the vector space isomorphisms

$$C^p(\underline{g}, M) \cong M \otimes_F \bigwedge^p \underline{g}^*, \qquad C^p(\underline{g}, M)/Z^p(\underline{g}, M) \cong B^{p+1}(\underline{g}, M)$$

(where $\underline{g}^*$ is the dual of $\underline{g}$ and $\bigwedge^p \underline{g}^*$ the vector space of homogeneous elements of degree $p$ of the Grassmann algebra over $\underline{g}^*$) one has

$$\binom{n}{p} \dim M = \dim Z^p(\underline{g}, M) + \dim B^{p+1}(\underline{g}, M);$$

therefore,

$$\text{(i)} \quad \dim H^p(\underline{g}, M) = \dim Z^p(\underline{g}, M) + \dim Z^{p-1}(\underline{g}, M) - \binom{n}{p-1} \dim M,$$

$$\text{(ii)} \quad \dim H^p(\underline{g}, M) = \binom{n}{p} \dim M - \dim B^p(\underline{g}, M) - \dim B^{p+1}(\underline{g}, M).$$

Denote

$$\chi_p(\underline{g}, M) = \sum_{q=0}^{p} (-1)^q \dim H^q(\underline{g}, M)$$

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(partial sum of the Euler-Poincaré characteristic); by using $\sum_{q=0}^{p}(-1)^q\binom{n}{q-1} = (-1)^p\binom{n-1}{p-1}$ one obtains

$$\text{(iii)} \qquad \dim Z^p(\underline{g}, M) = (-1)^p \chi_p(\underline{g}, M) + \binom{n-1}{p-1} \dim M,$$

$$\text{(iv)} \qquad \dim B^{p+1}(\underline{g}, M) = (-1)^p \chi_p(\underline{g}, M) + \binom{n-1}{p} \dim M.$$

(REMARK. Since $C^{n+1}(\underline{g}, M) = (0)$ one has $\dim B^{n+1}(\underline{g}, M) = 0$, therefore (iv) gives for the case $p = n$: $0 = (-1)^n \chi_n(\underline{g}, M) + 0 \cdot \dim M$; thus $\chi_n(\underline{g}, M) = 0$. Goldberg obtains this result for $M = F$ [3].)

1.3. Let $\underline{g}$ and $\underline{g}'$ be two Lie algebras of dimension $n$ and $n'$ over $F$, $\rho$ and $\rho'$ two representations of $\underline{g}$ and $\underline{g}'$ into a vector space $M$, $\phi\colon \underline{g} \to \underline{g}'$ a Lie algebra morphism such that $\rho = \rho' \circ \phi$. Denote

$$\phi_p = {}^t\left(\overset{p}{\wedge}\phi\right) = \overset{p}{\wedge}{}^t\phi\colon C^p(\underline{g}', M) \to C^p(\underline{g}, M);$$

then, obviously, $\phi_{p+1} \circ d'_p = d_p \circ \phi_p$; since $\operatorname{Im}\phi_p = \wedge^p(\operatorname{Ker}\phi)^\perp$ it follows that

$$\phi_p(Z^p(\underline{g}', M)) \subset \overset{p}{\wedge}(\ker\phi)^\perp \cap Z^p(\underline{g}, M),$$

$$\phi_p(B^p(\underline{g}', M)) \subset \overset{p}{\wedge}(\ker\phi)^\perp \cap B^p(\underline{g}, M).$$

1.4. LEMMA. *With the above notations, if $\phi$ is onto then*

$$\text{(i)} \qquad \phi_p(Z^p(\underline{g}', M)) = \overset{p}{\wedge}(\ker\phi)^\perp \cap Z^p(\underline{g}, M),$$

$$\text{(ii)} \qquad \dim H^p(\underline{g}', M) \leqslant \dim H^p(\underline{g}, M) + \left(\binom{n}{p-1} - \binom{n'}{p-1}\right) \dim M,$$

$$\text{(iii)} \qquad \dim H^p(\underline{g}, M) \leqslant \dim H^p(\underline{g}', M) + \left(\binom{n}{p} - \binom{n'}{p}\right) \dim M.$$

PROOF. If $\phi$ is onto then $\phi_p$ is one-to-one; let $f \in \operatorname{Im}\phi_p \cap Z^p(\underline{g}, M)$. One can write $f = \phi_p f'$ with $f' \in C^p(\underline{g}', M)$; then $0 = d_p f = d_p\phi_p f' = \phi_{p+1} d'_p f'$; therefore $d'_p f' = 0$, i.e. $f' \in Z^p(\underline{g}', M)$, which proves (i).

Since $\phi_p$ is one-to-one, we have

$$\dim Z^p(\underline{g}', M) = \dim \phi_p(Z^p(\underline{g}', M)) \leqslant \dim Z^p(\underline{g}, M);$$

thus

$$\dim H^p(\underline{g}', M) \leqslant \dim Z^p(\underline{g}, M) + \dim Z^{p-1}(\underline{g}, M) - \binom{n'}{p-1} \dim M$$

(by 1.2(i)), which proves (ii) (by 1.2(i) again).

By considering $B^p$ instead of $Z^p$, (iii) is obtained in the same way.

1.5. REMARKS. (i) The inclusion $\phi_p(B^p(\underline{g}', M)) \subset \wedge^p(\ker\phi)^\perp \cap B^p(\underline{g}, M)$ may be strict even with $\phi$ onto. For example let $\underline{g} = \mathfrak{H}_m$ be the Heisenberg Lie algebra of dimension $2m+1 \geqslant 5$ (2.2), $\underline{g}' = Fe'_1 \oplus \cdots \oplus Fe^*_{2m}$ the abelian Lie algebra of dimension $2m$ and $\phi\colon \mathfrak{H}_m \to \underline{g}'$, $e_i \mapsto e_i'$, $i \neq 0$, $e_0 \mapsto 0$. One has (by (2.2))

$$B^p(\mathfrak{H}_m, M) \subset Z^p(\mathfrak{H}_m, M) = \overset{p}{\wedge}(e^{*1},\dots,e^{*2m}) = \overset{p}{\wedge}(Fe_0)^{\perp} = \overset{p}{\wedge}(\ker\phi)^{\perp};$$

thus

$$B^p(\mathfrak{H}_m, M) \cap \overset{p}{\wedge}(\ker\phi)^{\perp} = B^p(\mathfrak{H}_m, M);$$

since $\phi_p(B^p(\underline{g}', M)) = \phi_p((0)) = (0)$ and $B^p(\mathfrak{H}_m, M) \cong \wedge^p(e^{*1},\dots,e^{*2m})$ (by (2.2)), the inclusion cannot be an equality.

(ii) For the above example of $\mathfrak{H}_m$ and $\underline{g}'$, 1.4(ii) is an equality (by (2.2)). In low dimension there are several other examples realizing the equality and proving thus that the inequality cannot be improved.

**2. Cohomological spaces of Heisenberg Lie algebras.**

2.1. REMARK. By the Poincaré duality $H^p(\mathfrak{H}_m)$ and $H^{2m+1-p}(\mathfrak{H}_m)$ are canonically isomorphic; therefore, one has to study only $H^p(\mathfrak{H}_m)$ for $p \leqslant m$.

2.2. THEOREM. *Let $\mathfrak{H}_m = Fe_0 \oplus \cdots \oplus Fe_{2m}$ be the Heisenberg Lie algebra of dimension $2m+1$ over a commutative field $F$, i.e. a Lie algebra satisfying $[e_i, e_{i+m}] = e_0$ $\forall i = 1,\dots,m$ (all the other brackets are 0). Let $p \in \{0,\dots,m\}$ and denote by $\{e^{*0},\dots,e^{*2m}\}$ the dual basis of $\{e_0,\dots,e_{2m}\}$. Then*

(i) *the pth Betti number (i.e. $\dim H^p(\mathfrak{H}_m)$) is equal to $\binom{2m}{p} - \binom{2m}{p-2}$;*

(ii) *the space of cocycles of degree $p$ of $\mathfrak{H}_m$ with values in $F$ is equal to the vector space of homogeneous elements of degree $p$ of the Grassmann algebra over $(e^{*1},\dots,e^{*2m})$, i.e.*

$$Z^p(\mathfrak{H}_m) = \overset{p}{\wedge}(e^{*1},\dots,e^{*2m});$$

(iii) *the space of coboundaries of degree $p$ of $\mathfrak{H}_m$ with values in $F$ is isomorphic to the vector space of homogeneous elements of degree $p-2$ of the Grassmann algebra over $(e^{*1},\dots,e^{*2m})$, the isomorphism being given by the exterior product by $d_1e^{*0}$:*

$$\overset{p-2}{\wedge}(e^{*1},\dots,e^{*2m}) \xrightarrow{\sim} B^p(\mathfrak{H}_m), \qquad \gamma \mapsto d_1e^{*0} \wedge \gamma.$$

PROOF. We will use induction on $m \geqslant 1$. The case $m = 1$ is obvious.

Let $\underline{g}' = Fe_1' \oplus \cdots \oplus Fe_{2m}'$ be the abelian Lie algebra of dimension $2m$ and $\phi\colon \mathfrak{H}_m \to \underline{g}'$ the morphism defined by $\phi e_i = e_i'$ $\forall i = 1,\dots,2m$, $\phi e_0 = 0$. In the notation of 1.3 one takes $M = F$, $\rho = \rho' = 0$; then, by 1.4(ii),

$$\binom{2m}{p} \leqslant \dim H^p(\mathfrak{H}_m) + \binom{2m+1}{p-1} - \binom{2m}{p-1}$$

and therefore $\binom{2m}{p} - \binom{2m}{p-2} \leqslant \dim H^p(\mathfrak{H}_m)$.

Let $\mathfrak{H} = Fe_0 \oplus \cdots \oplus Fe_{2m-1}$ and $\mathfrak{H}_{m-1} = Fe_0 \oplus \cdots \oplus \widehat{Fe_m} \oplus \cdots \oplus Fe_{2m-1}$ be the ideals of $\mathfrak{H}_m$ such that $\mathfrak{H} = \mathfrak{H}_{m-1} \times Fe_m$ (direct product). By the theorem of

Künneth [6, Chapter II, §5, p. 80] one has

$$H^p(\mathfrak{H}_{m-1} \times Fe_m) \cong \sum_{i+j=p} H^i(\mathfrak{H}_{m-1}) \otimes H^j(Fe_m);$$

therefore $\dim H^p(\mathfrak{H}) = \dim H^p(\mathfrak{H}_{m-1}) + \dim H^{p-1}(\mathfrak{H}_{m-1})$. Since $\mathfrak{H}_{m-1}$ is a Heisenberg Lie algebra of dimension $2(m-1)+1$, the induction hypothesis gives $\dim H^p(\mathfrak{H}_{m-1}) = \binom{2m-2}{p} - \binom{2m-2}{p-2}$ for $p \leq m-1$ and thus $\dim H^p(\mathfrak{H}) = \binom{2m-1}{p} - \binom{2m-1}{p-2}$ for $p \leq m-1$. By Proposition 1 of [2] one has the following exact sequence (because $\mathfrak{H}$ is an ideal of codimension 1 in $\mathfrak{H}_m$):

$$\cdots \to H^{p-1}(\mathfrak{H}) \to H^p(\mathfrak{H}_m) \to H^p(\mathfrak{H}) \to H^p(\mathfrak{H}) \to H^{p+1}(\mathfrak{H}_m) \to \cdots$$

which is a special case of the Hochschild-Serre spectral sequence [5]. Therefore, $\dim H^p(\mathfrak{H}_m) \leq \dim H^p(\mathfrak{H}) + \dim H^{p-1}(\mathfrak{H})$ and thus $\dim H^p(\mathfrak{H}_m) \leq \binom{2m}{p} - \binom{2m}{p-2}$ for $p \leq m-1$. Since the opposite inequality has already been proved one has conclusion (i) for $p \leq m-1$.

For $p = m$, one has to look more closely at the exact sequence of Dixmier. We proved that for $p \leq m-1$,

$$\dim H^p(\mathfrak{H}_m) = \dim H^p(\mathfrak{H}) + \dim H^{p-1}(\mathfrak{H});$$

therefore, the long exact sequence splits into small exact sequences

$$\begin{gathered}
(0) \to H^0(\mathfrak{H}_m) \to H^0(\mathfrak{H}) \to (0), \\
(0) \to H^0(\mathfrak{H}) \to H^1(\mathfrak{H}_m) \to H^1(\mathfrak{H}) \to (0), \\
(0) \to H^1(\mathfrak{H}) \to H^2(\mathfrak{H}_m) \to H^2(\mathfrak{H}) \to (0), \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(0) \to H^{m-2}(\mathfrak{H}) \to H^{m-1}(\mathfrak{H}_m) \to H^{m-1}(\mathfrak{H}) \to (0), \\
(0) \to H^{m-1}(\mathfrak{H}) \to H^m(\mathfrak{H}_m) \to H^m(\mathfrak{H}) \xrightarrow{u} H^m(\mathfrak{H}).
\end{gathered}$$

One then has $\dim H^m(\mathfrak{H}_m) = \dim H^{m-1}(\mathfrak{H}) + \dim \operatorname{Ker} u$. Let $\alpha \in H^m(\mathfrak{H}_{m-1})$ and $\beta \in H^{m-1}(\mathfrak{H}_{m-1})$; one has

$$\alpha \wedge \beta = c(\alpha, \beta) e^{*0} \wedge \cdots \wedge \hat{e}^{*m} \wedge \cdots \wedge e^{*2m-1} \quad \text{with } c(\alpha, \beta) \in F.$$

The duality of Poincaré asserts that

$$H^m(\mathfrak{H}_{m-1}) \times H^{m-1}(\mathfrak{H}_{m-1}) \to F, \qquad (\alpha, \beta) \to c(\alpha, \beta)$$

is a nondegenerate bilinear form; therefore $H^m(\mathfrak{H}_{m-1})$ is a quotient of

$$\bigwedge^m \left(e^{*0} \cdots \hat{e}^{*m} \cdots e^{*2m-1}\right)$$

(since $H^{m-1}(\mathfrak{H}_{m-1})$ is a quotient of $\bigwedge^{m-1}(e^{*1} \cdots \hat{e}^{*m} \cdots e^{*2m-1})$). On the other hand, by the theorem of Künneth,

$$H^m(\mathfrak{H}) = H^m(\mathfrak{H}_{m-1}) + H^{m-1}(\mathfrak{H}_{m-1}) \wedge e^{*m}.$$

Now $u$ is the action of $e_{2m}$ on $H^m(\mathfrak{H})$ [2, Proposition 1]; since $[e_m e_{2m}] = e_0$, $[e_i e_{2m}] = 0$, $i \neq m$, one has $e_{2m} \cdot e^{*0} = e^{*m}$, $e_{2m} \cdot e^{*i} = 0$, $i \neq m$; therefore

$$\operatorname{Im} u = \operatorname{Ker} u = H^{m-1}(\mathfrak{H}_{m-1}) \wedge e^{*m};$$

thus

$$\dim \operatorname{Ker} u = \dim H^{m-1}(\mathfrak{H}_{m-1}) = \binom{2m-2}{m-1} - \binom{2m-2}{m-3}$$

(induction). We then have

$$\begin{aligned}\dim H^m(\mathfrak{H}_m) &= \binom{2m-1}{m-1} - \binom{2m-1}{m-3} + \binom{2m-2}{m-1} - \binom{2m-2}{m-3} \\ &= \binom{2m}{m} - \binom{2m}{m-2}.\end{aligned}$$

This proves conclusion (i) for $p = m$.

By a simple computation one has $(-1)^p \chi_p(\mathfrak{H}_m, F) = \binom{2m}{p} - \binom{2m}{p-1}$ and by 1.2(iii), another computation gives $\dim Z^p(\mathfrak{H}_m) = \binom{2m}{p}$. From 1.3 it follows that $\phi_p(Z^p(g')) \subset Z^p(\mathfrak{H}_m)$. Since $Z^p(g') = \wedge^p(e'^{*1},\ldots,e'^{*2m})$, we get $\phi_p(Z^p(g')) = \wedge^p(e^{*1},\ldots,e^{*2m})$, therefore $\wedge^p(e^{*1},\ldots,e^{*2m}) \subset Z^p(\mathfrak{H}_m)$. Conclusion (ii) then follows from $\dim Z^p(\mathfrak{H}_m) = \binom{2m}{p}$.

Now

$$B^p(\mathfrak{H}_m) = \left\{ d_{p-1}\alpha;\ \alpha \in \bigwedge^{p-1}(e^{*0},\ldots,e^{*2m}) \right\}.$$

Any $\alpha \in \wedge^{p-1}(e^{*0},\ldots,e^{*2m})$ can be written $\alpha = \beta + e^{*0} \wedge \gamma$ with $\beta \in \wedge^{p-1}(e^{*1},\ldots,e^{*2m})$ $(= Z^{p-1}(\mathfrak{H}_m)$ by (ii)) and $\gamma \in \wedge^{p-2}(e^{*1},\ldots,e^{*2m})$ $(= Z^{p-2}(\mathfrak{H}_m)$ by (ii)); thus $d_{p-1}\alpha = 0 + d_1 e^{*0} \wedge \gamma - e^{*0} \wedge 0$ (recall that $d_p$ is an antiderivation); therefore

$$B^p(\mathfrak{H}_m) = \left\{ d_1 e^{*0} \wedge \gamma;\ \gamma \in \bigwedge^{p-2}(e^{*1},\ldots,e^{*2m}) \right\}.$$

Let $\psi\colon \wedge^{p-2}(e^{*1},\ldots,e^{*2m}) \to B^p(\mathfrak{H}_m)$, $\gamma \mapsto d_1 e^{*0} \wedge \gamma$; by the above result, $\psi$ is onto. By (i) and (ii), $\dim B^p(\mathfrak{H}_m) = \binom{2m}{p-2}$; thus $\dim B^p(\mathfrak{H}_m) = \dim \wedge^{p-2}(e^{*1},\ldots,e^{*2m})$, which proves that $\psi$ is an isomorphism.

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