

GEOMETRIC REALIZATION OF A FINITE SUBGROUP OF $\pi_0 \epsilon(M)$. II

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ABSTRACT. Let M be a closed aspherical manifold with a virtually 2-step nilpotent fundamental group. Then any finite group G of homotopy classes of self-homotopy equivalences of M can be realized as an effective group of self-homeomorphisms of M if and only if there exists a group extension E of π by G realizing $G \rightarrow \text{Out } \pi_1 M$ so that $C_E(\pi)$, the centralizer of π in E , is torsion-free. If this is the case, the action (G, M) is equivalent to an affine action (G, M') on a complete affinely flat manifold homeomorphic to M . This generalizes the same result for flat Riemannian manifolds.

Let M be a closed aspherical manifold, $\epsilon(M)$ the group of homotopy equivalences of M into itself. Any $f \in \epsilon(M)$ induces an isomorphism $f_*: \pi_1(M, x) \rightarrow \pi_1(M, f(x))$. By choosing a path ω from x to $f(x)$, we get an automorphism f_*^ω of $\pi_1(M, x)$, mapping $[\tau]$ to $[\omega^{-1}(f \circ \tau)\omega]$ for any loop τ based at x . A different choice of τ alters f_*^ω by an inner automorphism of $\pi_1(M, x)$. Therefore, we obtain a natural surjective homomorphism $\Psi: \epsilon(M) \rightarrow \text{Out } \pi_1 M$, where $\text{Out } \pi_1 M = \text{Aut } \pi_1 M / \text{Inn } \pi_1 M$, the group of automorphism classes. Of course, $\Psi(f) = [f_*^\omega]$ for any ω as above. Since we are considering only $\pi_1(M, x)$, we need not specify the base point, so we write $\pi_1 M$ instead of $\pi_1(M, x)$. It is well known that for a closed aspherical manifold M , the kernel of Ψ is $\epsilon_0(M)$, the group of self-homotopy equivalences of M homotopic to the identity. This implies that Ψ factors through $\pi_0 \epsilon(M)$ so that

$$\begin{array}{ccc} & \epsilon(M) & \\ \swarrow & & \searrow \Psi \\ \pi_0 \epsilon(M) & \xrightarrow[\cong]{\Psi_*} & \text{Out}(\pi_1 M) \end{array}$$

commutes. A pair of groups G, π together with a homomorphism $\phi: G \rightarrow \text{Out } \pi$ is called an *abstract kernel*, and is denoted by (G, π, ϕ) . Therefore, an injective abstract kernel $(G, \pi_1 M, \phi)$ for a closed aspherical manifold M is really a group of homotopy classes of self-homotopy equivalences of M .

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Restricting Ψ to $H(M)$, the group of self-homeomorphisms of M , we have a homomorphism $\Psi: H(M) \rightarrow \text{Out } \pi_1 M$. This is surjective since M is a closed aspherical manifold. A *geometric realization* of an abstract kernel $\phi: G \rightarrow \text{Out } \pi_1 M$ is a homomorphism $\hat{\phi}: G \rightarrow H(M)$ so that $\Psi \circ \hat{\phi} = \phi$. In this paper we will require $\hat{\phi}$ to be injective so that the action of G on M is effective. Therefore, when the abstract kernel is injective, the problem reduces to the problem of geometric realization of a group of homotopy classes of self-homotopy equivalences as a group of self-homeomorphisms.

Suppose we have a G -action. Let $\mathcal{L}(G, M)$ be the group of all liftings of elements of G to \tilde{M} , the universal covering of M . Then

$$1 \rightarrow \pi_1 M \rightarrow \mathcal{L}(G, M) \rightarrow G \rightarrow 1$$

is exact and admissible (i.e., $C_{\mathcal{L}(G, M)}(\pi_1 M)$, the centralizer of $\pi_1 M$ in $\mathcal{L}(G, M)$, is torsion-free). See [LR1, Lemma 1]. Therefore, in order that an abstract kernel $(G, \pi_1 M, \phi)$ for a closed aspherical manifold M be realized geometrically, it is necessary that the abstract kernel have an extension which is admissible. In certain cases, this condition is also sufficient when G is finite. For example, it is true for closed flat Riemannian manifolds (see [LR1; ZZ] when ϕ is injective), for certain Seifert fibered spaces [R], and for Seifert relatives of compact flat Riemannian manifolds [L2]. In this paper we will verify that this is also the case for closed aspherical manifolds with virtually 2-step nilpotent fundamental groups. In a subsequent joint paper with Frank Raymond we shall generalize this to more general Seifert fibered spaces.

In [L3] the following is proved.

FACT 1 [L3]. A closed aspherical manifold M of dimension $\neq 3$ or 4 with a virtually 2-step nilpotent fundamental group is homeomorphic to a complete affinely flat manifold M' .

We now state our main theorem.

THEOREM. *Let M^m be a closed aspherical manifold ($m \neq 3, 4$) with a virtually 2-step nilpotent fundamental group. A finite abstract kernel $\phi: G \rightarrow \text{Out } \pi_1 M$ can be realized as an effective group of self-homeomorphisms of M if and only if it admits an admissible extension. In fact, (G, M) is equivalent to an affine action (G, M') on a closed affinely flat manifold M' homeomorphic to M .*

PROOF OF THE THEOREM. We will need the following facts.

FACT 2 [FH]. Let N^m ($m \neq 3, 4$) be a closed connected infranilmanifold and M^n be an aspherical manifold with $\pi_1(M^n)$ isomorphic to $\pi_1(N^m)$. Then M^m and N^m are homeomorphic.

FACT 3 [L2]. Let Q be a virtually free abelian group (of rank n) which has a representation $\phi \times \rho: Q \rightarrow \text{GL}(k, \mathbf{Z}) \times E(n)$ such that $\text{image}(\phi)$ is finite, $\text{image}(\rho)$ is a crystallographic group (of rank n). Then for any extension E of \mathbf{Z}^k by Q

realizing ϕ , there exists a homomorphism $f: E \rightarrow L \circ Q_1 \subset A(k+n)$ so that

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^k & \rightarrow & E & \xrightarrow{p} & Q & \rightarrow & 1 \\ & & \downarrow & & \downarrow f & & \downarrow \phi \times \rho & & \\ 1 & \rightarrow & L & \rightarrow & L \circ Q_1 & \rightarrow & Q_1 & \rightarrow & 1 \end{array}$$

commutes. Moreover, f is injective if and only if $E_K = \ker((\phi \times \rho) \circ p)$ is torsion-free.

In the above, $E(n)$ denotes the group of rigid motions on \mathbf{R}^n , $A(k+n)$ the group of affine motions on \mathbf{R}^{k+n} , $L = L(\mathbf{R}^n, \mathbf{R}^k)$ is the group of all affine maps of \mathbf{R}^n into \mathbf{R}^k and $Q_1 = \text{image}(\phi \times \rho)$. A discrete subgroup P of $E(n)$ with \mathbf{R}^n/P compact is called a crystallographic group of rank n .

Suppose the abstract kernel $\phi: G \rightarrow \text{Out } \pi$ ($\pi = \pi_1 M$) admits an admissible extension E so that $1 \rightarrow \pi \rightarrow E \rightarrow G \rightarrow 1$ is exact and $C_E(\pi)$ is torsion-free. Let N be the nilradical of π , that is, N is the maximal nilpotent normal subgroup of π . Then N is characteristic in π . Since E is also virtually 2-step nilpotent, N is an extension of a torsion-free nilpotent subgroup by a finite group. In addition, N is torsion-free. This implies that N itself is 2-step nilpotent. The center $z(N)$ of N is normal in E . Since $E/z(N)$ contains a free abelian subgroup $N/z(N)$, it is virtually free abelian. Certainly the natural homomorphism $\phi: E/z(N) \rightarrow \text{Aut}(z(N)) = \text{GL}(k, \mathbf{Z})$ ($k = \text{rank of } z(N)$) has a finite image. There exists a homomorphism $\rho: E/z(N) \rightarrow E(n)$ ($n = m - k$) with crystallographic group as image. See [LR1] for the proof. Clearly $\ker(\rho)$ is finite. Let E_K be the preimage of $\ker(\phi \times \rho)$ under the natural homomorphism $E \rightarrow E/z(N)$. We shall prove that E_K is torsion-free.

First we show $E_K \subset C_E(N)$. Suppose $x \in E_K$. Then $[x, N] \subset z(N)$, $[x, z(N)] = 1$ and $x^t \in z(N)$ for some $t > 0$. Here, $[x, y] = xyx^{-1}y^{-1}$. Let X be the subgroup of E generated by x . The first two conditions imply that $X \cdot N$ is again 2-step nilpotent. Since $x^t \in z(N)$, $[x^t, N] = 1$. $[x^t, \alpha] = [x, \alpha]^t$ for all $\alpha \in N$ since $X \cdot N$ is 2-step nilpotent. Note that $[x, \alpha] \in N \subset \pi$. Therefore $[x, \alpha]^t = [x^t, \alpha] = 1$ with $[x, \alpha] \in \pi$. Since π is torsion free, $[x, \alpha] = 1$. That is, $x \in C_E(N)$.

Next we show that $\text{Tor}(C_E(N)) = \text{Tor}(C_E(\pi))$. Suppose θ is an automorphism of π such that $\theta|_N$ is the identity. We claim that $\theta(\sigma) \cdot \sigma^{-1} \in C_E(N)$ for all $\sigma \in \pi$. For any $\alpha \in N$,

$$\sigma \cdot \alpha \cdot \sigma^{-1} = \theta(\sigma \cdot \alpha \cdot \sigma^{-1}) = \theta(\sigma) \cdot \theta(\alpha) \cdot \theta(\sigma)^{-1} = \theta(\sigma) \cdot \alpha \cdot \theta(\sigma)^{-1}.$$

This implies that $\theta(\sigma) \cdot \sigma^{-1} \in C_\pi(N)$, since $C_\pi(N)$ is characteristic in π . Now let $c \in C_E(N)$ be an element of finite order, say, $c^r = 1$ for some $r > 0$. We apply the results above to the automorphism $\theta = \text{conjugation by } c$. Certainly θ induces the identity on N , for $c \in C_E(N)$. Therefore $[c, \sigma] = \theta(\sigma) \cdot \sigma^{-1} \in C_\pi(N)$. We claim that $C_\pi(N) = z(N)$. Since $C_\pi(N)$ is a torsion-free central extension of $z(N)$ by a finite group, it is again free abelian of rank $k = \text{rank } z(N)$ (see [LR1, Fact 2]). Therefore, $C_\pi(N) \cdot N$ is a nilpotent normal subgroup of π containing N . However, N is the nilradical of π , which implies $C_\pi(N) \subset N$. For any $\sigma \in \pi$,

$$[c^r, \sigma] = [c, \sigma]^{c^{r-1}} \cdots [c, \sigma]^{c^2} \cdot [c, \sigma]^c \cdot [c, \sigma]$$

where $x^y = yxy^{-1}$. Since $[x, \sigma] \in N$ and $c \in C_E(N)$, $c[c, \sigma]c^{-1} = [c, \sigma]$. Therefore, $1 = [c^r, \sigma] = [c, \sigma]^r$. As N is torsion-free $[c, \sigma] = 1$, i.e. $c \in C_E(\pi)$. Consequently $\text{Tor}(E_K) \subset \text{Tor}(C_E(N)) = \text{Tor}(C_E(\pi))$ and hence, E_K is torsion-free (since so is $C_E(\pi)$).

Thus we have $\phi: E/z(N) \rightarrow \text{GL}(k, Z)$ with a finite image, $\rho: E/z(N) \rightarrow E(n)$ with a finite kernel and with a crystallographic group (of rank n) as image. Furthermore, we have just proved that E_K = preimage of $\text{kernel}(\phi \times \rho)$ under $E \rightarrow E/z(N)$ is torsion-free. Applying Fact 3, we obtain an embedding f of E into $A(m)$. Via f , π acts on \mathbf{R}^m as a covering transformation yielding $M' = \mathbf{R}^m/f(\pi)$, a compact complete affinely flat manifold. Furthermore, $G = E/\pi$ acts on M' as a group of affine diffeomorphisms. By Fact 2, M and M' are homeomorphic. Then the G -action on M' can be pulled back to one on M via this homeomorphism. Q.E.D.

EXAMPLE. A finite subgroup G of $\text{Out } \pi_1 M$ cannot be realized as a group of affine actions on M in general. Let $\pi = \langle t_1, t_2, t'_3 \rangle$ be the subgroup of $A(3)$, $t_1 = (I, e_1)$, $t_2 = (I, e_2)$ and $t'_3 = (T, e_3)$ with $T = I + E_{21}$. Consider the subgroup Z_2 of $\text{Out } \pi$ generated by the automorphism $t_1 \rightarrow t_1^{-1}$, $t_2 \rightarrow t'_3$, $t'_3 \rightarrow t_2$. This $Z_2 \subset \text{Out } \pi$ can be realized as a group of homeomorphisms but not as a group of affine maps. For if the latter were true, then the derivatives of t_2 and t'_3 would be similar 3×3 matrices. But by construction these derivations are I and T respectively, and these matrices are definitely not similar.

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