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PROOF OF SCOTT'S CONJECTURE

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ABSTRACT. We give a proof of Conjecture 7 in [**2**, p. 155] first stated in 1881 by R. F. Scott [**4**]. It reads as follows:

CONJECTURE 7 (R. F. SCOTT). Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be the distinct roots of $x^n - 1 = 0$ and of $y^n + 1 = 0$, respectively. Let A be the $n \times n$ matrix whose (i, j) entry is $1/(x_i - y_j)$, $i, j = 1, \dots, n$. Then

$$|\mathrm{per}(A)| = \begin{cases} n(1 \cdot 3 \cdot 5 \cdots (n-2))^2/2^n, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$$

Actually, our proof gives more, namely an explicit expression for $\mathrm{per}(A)$ (see Theorem 2.1).

1. Preliminary results. Let A be an $n \times n$ matrix (a_{ij}). The permanent of A is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group over the set $\{1, \dots, n\}$.

We first quote two results which we shall use in the course of the proof.

THEOREM 1.1 (BORCHARDT [**1**]). *Let A be an $n \times n$ matrix whose (i, j) entry is $(s_i - t_j)^{-1}$, where s_i, t_j are complex numbers. Then*

$$\det(A)\mathrm{per}(A) = \det(A^{(2)}),$$

where $A^{(2)}$ has entries $(s_i - t_j)^{-2}$.

For the proof see e.g. [**2**, p. 6]. .

The next result which we need concerns circulants. For given numbers $a_0, \dots, a_{n-1}$ the circulant $C(a_0, \dots, a_{n-1}) = (c_{ij})$ is a $n \times n$ matrix, whose entries are given by

$$c_{ij} = \begin{cases} a_{j-i}, & \text{if } i \leqslant j, \\ a_{n+j-i}, & \text{if } i > j. \end{cases}$$

THEOREM 1.2. *The determinant of the circulant is given by*

$$\det C(a_0, \dots, a_{n-1}) = \prod_{k=1}^{n} f(x_k),$$

where $f(t) = \sum_{i=0}^{n-1} a_i t^i$, and x_k are the nth roots of unity.

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This can be proved easily by multiplying $\det C(a_0, \ldots, a_{n-1})$ by the Vandermonde determinant $V(x_1, \ldots, x_n)$ (cf. [3, pp. 442, 445]).

We shall also need the following properties of the set $H = \{h \in \mathbf{C}\colon h^n = -1\}$ of the nth roots of -1.

PROPOSITION 1.3. (i) $\sum_{h \in H}(1 - h)^{-1} = n/2$.
(ii) $\sum_{h \in H}(1 - h)^{-2} = n(2 - n)/4$.

PROOF. Let $h \in H$. Put $x = (1 - h)^{-1}$. Then $h = (x - 1)/x$ and the following equivalences hold

$$h^n = -1 \Leftrightarrow (x-1)^n + x^n = 0 \Leftrightarrow 2x^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} x^{n-k} = 0. \tag{1}$$

This enables us to interpret $S = \{x = (1 - h)^{-1}\colon h \in H\}$ as the set of solutions of the equation (1).

By the Viète formulas we obtain

$$\sum_{x \in S} x = -(-1)^1 \binom{n}{1} / 2 = n/2,$$

so that (i) is proved.

To prove (ii) observe that

$$\sum_{x \in S} x^2 = \Big(\sum_{x \in S} x \Big)^2 - \Big(\sum_{x \neq y} xy \Big).$$

Again by Viète formula we obtain

$$\sum_{x \in S} x^2 = (n/2)^2 - 2(-1)^2 \binom{n}{2} / 2 = n(2 - n)/4,$$

thus proving (ii).

2. The main result.

THEOREM 2.1. *Let A be the $n \times n$ matrix whose (i, j) entry is $(x_i - y_j)^{-1}$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the distinct roots of $x^n - 1 = 0$ and $y^n + 1 = 0$ respectively. Then*

$$\operatorname{per}(A) = \begin{cases} (-1)^{[n/2]} n (1 \cdot 3 \cdot 5 \cdots (n-2))^2 / 2^n, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$$

PROOF. By Theorem 1.1 we can write

$$D_1 \cdot \operatorname{per}(A) = D_2, \tag{1}$$

where D_ν, $\nu = 1, 2$, stands for $\det((x_i - y_j)^{-\nu})$.

In order to obtain an evaluation of D_ν's we first make the following observations.

(a) Let ε be any nth root of -1. Then the map $g \mapsto \varepsilon g$ from $G = \{g\colon g^n = 1\}$ to $H = \{h\colon h^n = -1\}$ is a bijection.

(b) From the definition of the permanent it is readily seen that

$$\operatorname{per}\big((x_{\sigma(i)} - y_{\tau(j)})^{-1}\big) = \operatorname{per}\big((x_i - y_j)^{-1}\big),$$

for any $\sigma, \tau \in S_n$.

Therefore, without loss of generality we can assume that the nth roots of 1 and -1 are of the form

$$x_i = \alpha^i, \qquad y_j = \varepsilon\alpha^j, \qquad i, j = 1, \ldots, n,$$

where α is an arbitrary primitive nth root of 1.

By factoring out $x_i^{-\nu}$ from the ith row ($i = 1, \ldots, n$) we get

$$\text{(2)} \quad D_\nu = \left(\prod_{i=1}^{n} x_i\right)^{-\nu} \cdot \det\left((1 - \varepsilon\alpha^{j-i})^{-\nu}\right) = (-1)^{(n-1)\nu} \cdot \det C(a_0^\nu, \ldots, a_{n-1}^\nu)$$

where $a_k = (1 - \varepsilon\alpha^k)^{-1}$, $k = 0, 1, \ldots, n-1$.

Applying Theorem 1.2 to the circulant $C_\nu = C(a_0^\nu, \ldots, a_{n-1}^\nu)$ it follows

$$\det C_\nu = \prod_{k=1}^{n} f_\nu(x_k),$$

where

$$\text{(3)} \qquad f_\nu(t) = \sum_{i=0}^{n-1} a_i^\nu t^i.$$

For fixed k, $1 \leqslant k \leqslant n$, we compute

$$\begin{aligned} f_\nu(x_k) &= \sum_{i=0}^{n-1} (1 - \varepsilon\alpha^i)^{-\nu}(\alpha^k)^i = \sum_{i=0}^{n-1} (1 - \varepsilon\alpha^i)^{-\nu}(\alpha^i)^k \\ &= \varepsilon^{-k} \sum_{h \in H} (1 - h)^{-\nu} h^k \quad (\text{writing } h = 1 - (1 - h)) \\ &= \varepsilon^{-k} \sum_{h \in H} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1 - h)^{j-\nu}. \end{aligned}$$

Splitting the second sum into two parts given by $j \leqslant \nu - 1$ and $j \geqslant \nu$ and using the fact that

$$\sum_{h \in H} h^t = \begin{cases} n, & \text{if } t = 0, \\ 0, & \text{if } 0 < t < n, \end{cases}$$

we obtain

$$f_\nu(x_k) = \varepsilon^{-k} \sum_{h \in H} \left[\sum_{j=0}^{\nu-1} (-1)^j \binom{k}{j} (1 - h)^{j-\nu} + \sum_{j=\nu}^{k} (-1)^j \binom{k}{j}\right].$$

Since $\sum_{j=\nu}^{k} (-1)^j \binom{k}{j} = -\sum_{j=0}^{\nu-1} (-1)^j \binom{k}{j}$, it follows

$$f_\nu(x_k) = \varepsilon^{-k} \sum_{j=0}^{\nu-1} (-1)^j \binom{k}{j} \left[\sum_{h \in H} (1 - h)^{j-\nu} - n\right].$$

Now, using Proposition 1.3, we get

$$f_\nu(x_k) = \varepsilon^{-k} \begin{cases} -n/2, & \text{if } \nu = 1, \\ \left(\dfrac{n(2-n)}{4} - n\right) - k\left(\dfrac{n}{2} - n\right), & \text{if } \nu = 2, \end{cases}$$

or

$$f_\nu(x_k) = \varepsilon^{-k}\begin{cases} -n/2, & \text{if } \nu = 1, \\ -n(n-2k+2)/4, & \text{if } \nu = 2. \end{cases} \tag{4}$$

Substituting (4) in (3) and (2) we obtain

$$D_\nu = \begin{cases} -\left(\frac{n}{2}\right)^n \prod_{k=1}^{n} \varepsilon^{-k}, & \text{if } \nu = 1, \\ \left(-\frac{n}{4}\right)^n \prod_{k=1}^{n} \varepsilon^{-k}(n-2k+2), & \text{if } \nu = 2. \end{cases} \tag{5}$$

From (5) it is clear that $D_1 \neq 0$. Hence, by (1),

$$\begin{aligned} \operatorname{per}(A) &= D_2/D_1 \\ &= (-1)^{n-1}\prod_{k=1}^{n}(n-2k+2)/2^n \\ &= \begin{cases} (-1)^{[n/2]} n(1\cdot 3\cdots(n-2))^2/2^n, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases} \end{aligned}$$

which completes the proof.

As a corollary we have the following identity:

COROLLARY 2.2.

$$\sum_{\sigma\in S_n} \frac{1}{\prod_{k=1}^{n}\sin(\pi(2k-2\sigma(k)+1)/2n)} = \begin{cases} n(1\cdot 3\cdots(n-2))^2, & \textit{if } n \textit{ is odd}, \\ 0, & \textit{if } n \textit{ is even}. \end{cases}$$

PROOF. Put $\alpha = \exp(2\pi i/n)$, $\varepsilon = \exp(\pi i/n)$, $i = \sqrt{-1}$. Then for the matrix A from Theorem 2.1 we obtain

$$\operatorname{per} A = (-1)^{n-1}\operatorname{per}\left((1-\varepsilon\cdot\alpha^{k-j})^{-1}\right) \qquad \text{(cf. (2))}.$$

Since

$$\frac{1}{1-\varepsilon\alpha^{k-j}} = \frac{1}{\sin(\pi(2(k-j)+1)/2n)}\cdot\frac{i}{2}\exp\left(\frac{2(j-k)-1}{2n}\pi i\right)$$

and

$$\prod_{k=1}^{n}\exp\left(\frac{2(\sigma(k)-k)-1}{2n}\pi i\right) = -i,$$

for any $\sigma \in S_n$, we get

$$\operatorname{per} A = (-1)^n\frac{i^{n+1}}{2^n}\operatorname{per}\left(\frac{1}{\sin(\pi(2(k-j)+1)/2n)}\right).$$

Now by Theorem 2.1 the result follows, because the last permanent is precisely equal to the left-hand side of the desired identity.

NOTE ADDED IN PROOF. After this article was written (November, 1980) we learnt that Minc (Linear Algebra and Its Applications **28** (1979), 141 – 153, and Kittappa

(Linear and Multilinear Algebra **10** (1981), 75 – 82) also proved Scott's conjecture. We hope that our proof, which is entirely algebraic and self-contained, is also shorter and more elementary.

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