PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

PRINCIPAL HOMOGENEOUS SPACES OVER HENSEL RINGS

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ABSTRACT. We prove that if $(A, \underline{a})$ is a Hensel couple and G is an affine, smooth group scheme over A then $H^1_{\mathrm{et}}(A, G) = H^1_{\mathrm{et}}(A/\underline{a}, G/\underline{a}G)$.

Introduction. In this paper we prove the following

THEOREM 1. *Let $(A, \underline{a})$ be a Hensel couple and let G be an affine, smooth group scheme over A, i.e. a functor G: (A-algebras) $\to$ (groups) which is represented by a smooth A-algebra (also denoted G). Then the canonical map $H^1_{\mathrm{et}}(A, G) \to H^1_{\mathrm{et}}(A/\underline{a}, G/\underline{a}G)$ is bijective.*

This theorem generalizes a result known when A is local and $\underline{a}$ is the maximal ideal (see [**4**, Theorem 11.7]).

The injectivity of the above map was proved in [**8**]. We also use the main result of [**3**].

Recall that $H^1_{\mathrm{et}}(A, G)$ classifies the isomorphism classes of principal homogeneous spaces in the étale topology over A under G (PHS for short, "torseurs" in French terminology).

In §§1 and 2 we give some preliminaries. In §3 we prove Theorem 1 in the case when A is an AIC ring, and we conclude the proof in §4, where some corollaries are also given.

As for the notion of Hensel couple, see [**2**]; all the notions about group schemes and cohomology can be found in [**1** and **6**].

1. In this section we give the definition and some properties of absolutely integrally closed rings that we will use later (see also [**3**, 2.B]).

A ring A is said to be *absolutely integrally closed* (AIC for short) if it satisfies one of the following equivalent conditions:

(a) every monic polynomial in $A[X]$ has a root in A;

(b) every monic polynomial in $A[X]$ splits into a product of linear factors.

PROPOSITION 1. *The following properties hold*:

(1) *Every ring of fractions and every quotient of an AIC ring is AIC.*

(2) *A local AIC ring is strict henselian.*

(3) *If A is an AIC ring and $\underline{a} \subset A$ is an ideal, then the henselization ${}^h(A, \underline{a})$ is AIC.*

Received by the editors January 20, 1982.

1980 *Mathematics Subject Classification*. Primary 13J15, 14F20; Secondary 14L15.

[1]This paper was done within the Group for Algebra and Geometry (GNSAGA) of the Italian National Research Council (CNR).

0002-9939/82/0000-0562/$ 02.00

PROOF. (1) and (2) are easy. (3) By [2, Theorem 6.1], ${}^h(A, \underline{a})$ is a direct limit of N-extensions of $(A, \underline{a})$; hence we reduce to the case of $B = A[x]_{1+(\underline{a},x)A[x]}$ where $A[x] = A[X]/(f)$ and f is an N-polynomial, i.e. $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ with $a_0 \in \underline{a}$, a_1 invertible modulo $\underline{a}$. We write $f = (X - b_1) \cdots (X - b_n)$, $b_i \in A$, and show that $B = \prod_{i=1}^n A_{1+(\underline{a},b_i)}$. In fact for every $i = 1,\ldots,n$ if we put $f = (X - b_i)g_i(X)$, we have $f'(X) = g_i(X) + (X - b_i)g_i'(X)$; but the image $f'(x)$ of $f'(X)$ in B is invertible since $f'(x) \in 1 + (\underline{a}, x)A[x]$.

Given any ring A we can construct a ring A^* as follows: let $\{f_i\}_{i \in I}$ be the set of monic polynomials in $A[X]$ and put $A_i = A[X]/(f_i)$; define $A_1 = \otimes_{i \in I} A_i$ and, inductively, $A_n = (A_{n-1})_1$; finally let $A^* = \varinjlim A_n$.

It is easy to see that the ring A^* is an AIC ring, faithfully flat and integral over A, and it is the direct limit of A-algebras which are free with finite rank as A-modules.

2. In this section we deduce from the main result of [3] a method of descent for modules and algebras.

Let $(A, \underline{a})$ be an H-couple. For every $f \in A$ we denote by ${}^hA_f$ the henselization of A_f with respect to $\underline{a}A_f$. Let $f_1,\ldots,f_n \in A$ such that $(f_1,\ldots,f_n) = A$. In [3, Theorem 1.11] we proved that if M_i are ${}^hA_{f_i}$-modules with isomorphisms

$$\phi_{ij}\colon M_i \otimes_{{}^hA_{f_i}} {}^hA_{f_if_j} \simeq M_j \otimes_{{}^hA_{f_j}} {}^hA_{f_if_j}$$

and the ϕ_{ij}'s satisfy the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on ${}^hA_{f_if_jf_k}$, then there is a unique A-module M such that $M \otimes_A {}^hA_{f_i} \simeq M_i$, $i = 1,\ldots,n$. This easily particular we apply this "henselian descent" to the following case. Let G be an affine group scheme of finite presentation over A. Recall that a principal homogeneous space for the étale topology over A under G is a G-space, i.e. an A-algebra H together with a G-action (see [1, II, 1.3.1]), such that $H \otimes_A S \simeq G \otimes_A S$ as $G \otimes_A S$-spaces over S for some faithfully flat étale homomorphism $A \to S$.

Now suppose we have PHS's H_i over ${}^hA_{f_i}$ and isomorphisms

$$\phi_{ij}\colon H_i \otimes_{{}^hA_{f_i}} {}^hA_{f_if_j} \simeq H_j \otimes_{{}^hA_{f_j}} {}^hA_{f_if_j}$$

as $G \otimes_A {}^hA_{f_if_j}$-spaces over ${}^hA_{f_if_j}$, such that the ϕ_{ij}'s satisfy the cocycle condition over ${}^hA_{f_if_jf_k}$; then there is a unique PHS H over A such that $H \otimes_A {}^hA_{f_i} \simeq H_i$ as $G \otimes_A {}^hA_{f_i}$-spaces over ${}^hA_{f_i}$. In fact from the above remark we can descend the algebras H_i and the $G \otimes_A {}^hA_{f_i}$-action and obtain a G-space H over A; moreover H is of finite presentation over A since the H_i's are of finite presentation over ${}^hA_{f_i}$; and, since $H \otimes_A S_i \simeq G \otimes_A S_i$ as $G \otimes_A S_i$-spaces over S_i for faithfully flat étale ${}^hA_{f_i}$-algebras S_i, we easily see that there exists a faithfully flat étale A-algebra S such that $H \otimes_A S \simeq G \otimes_A S$ as $G \otimes_A S$-spaces over S. Thus H is a PHS over A under G.

3. In this section we prove Theorem 1 in the case when A is an AIC ring. First of all we recall some known facts about cohomology.

PROPOSITION 2. *Let G be an affine smooth group scheme over A; then*

(a) *$H^1_{\mathrm{et}}(A, G) \simeq H^1_{\mathrm{fp}}(A, G)$, where* fp *denotes the faithfully flat finite presentation topology, and*

(b) *if $(A, \underline{a})$ is an H-couple, the canonical map $H^1_{\mathrm{et}}(A, G) \to H^1_{\mathrm{et}}(A/\underline{a}, G/\underline{a}G)$ is injective.*

PROOF. (a) See [4, Corollary 11.9].

(b) See [8, I, Theorem 2 and II, Proposition 2].

THEOREM 2. *Let $(A, \underline{a})$ be an H-couple and let G be an affine smooth group scheme over A. Suppose A is an AIC ring. Then*

$$H^1_{\text{et}}(A, G) \simeq H^1_{\text{et}}(A/\underline{a}, G/\underline{a}G).$$

PROOF. We have to prove only surjectivity. Let $\overline{H}$ be a PHS over $\overline{A} = A/\underline{a}$, and let $I \subset A$ be the set of elements $f \in A$ such that there exists a PHS $H(f)$ over ${}^hA_f$ for which $H(f)/\underline{a}H(f) \simeq \overline{H}_f$ as $G_f/\underline{a}G_f$-spaces. We want to prove that $I = A$. Let $\underline{p} \in \operatorname{Spec} A$; since $\overline{H} \otimes_A A_p$ is trivial by Proposition 1(2), there exists $f \notin \underline{p}$ such that $\overline{H}_f$ is trivial; hence $\overline{H}_f$ can be lifted to $G \otimes_A {}^hA_f$. It remains to prove that I is an ideal of A. Let $f, g \in I$; replacing A with ${}^hA_s$, $s = f + g$, we can suppose $(f, g) = A$. So we have PHS's $H(f)$, $H(g)$ over ${}^hA_f$, ${}^hA_g$, respectively, such that $H(f)/\underline{a}H(f) \simeq \overline{H}_f$, $H(g)/aH(g) \simeq \overline{H}_g$; if we consider $H(f) \otimes_{{}^hA_f} {}^hA_{fg}$ and $H(g) \otimes_{{}^hA_g} {}^hA_{fg}$ they are isomorphic PHS's over ${}^hA_{fg}$ because of Proposition 2(b). Then by the remark in §2 there exists a PHS H over A such that $H/\underline{a}H \simeq \overline{H}$.

4. In this section we conclude the proof of Theorem 1 and deduce some corollaries.

PROOF OF THEOREM 1. We have to prove the surjectivity of the map $H^1_{\text{et}}(A, G) \to H^1_{\text{et}}(\overline{A}, \overline{G})$, where $\overline{A} = A/\underline{a}$, $\overline{G} = G/\underline{a}G$. Let $\overline{H}$ be a PHS over $\overline{A}$ under $\overline{G}$ and consider the ring A^* defined in §1; $\overline{H} \otimes_{\overline{A}} A^*$ is a PHS over $A^*/\underline{a}A^*$ under $G \otimes_A A^*/\underline{a}A^*$ and hence by Theorem 2 it can be lifted to a PHS H^* over A^* such that $H^* \otimes_A \overline{A} \simeq \overline{H} \otimes_{\overline{A}} A^*$. Since A^* is the limit of A-algebras which are free with finite rank as A-modules, we can find such an algebra A' and a PHS H' over A' such that $H' \otimes_A \overline{A} \simeq \overline{H} \otimes_{\overline{A}} A'$.

Now we want to descend H' to a PHS H over A such that $H/\underline{a}H \simeq \overline{H}$. In order to do this consider the two extensions H_1, H_2 of H' to $A' \otimes_A A'$ and the three extensions H_{12}, H_{23}, H_{13} of H' to $A' \otimes_A A' \otimes_A A'$. Let us consider the three functors G_1, G_2, G_3: (A-algebras) $\to$ (sets) defined as follows: for every A-algebra B,

$$\begin{aligned}
G_1(B) &= \operatorname{Isom}(H' \otimes_A B, H' \otimes_A B) && \text{as } G \otimes_A A' \otimes_A B\text{-spaces over } A' \otimes_A B,\\
G_2(B) &= \operatorname{Isom}(H_1 \otimes_A B, H_2 \otimes_A B) && \text{as } G \otimes_A A' \otimes_A A' \otimes_A B\text{-spaces}\\
&&& \text{over } A' \otimes_A A' \otimes_A B,\\
G_3(B) &= \operatorname{Isom}(H_{23} \otimes_A B, H_{12} \otimes_A B) && \text{as } G \otimes_A A' \otimes_A A' \otimes_A A' \otimes_A B\text{-spaces}\\
&&& \text{over } A' \otimes_A A' \otimes_A A' \otimes_A B.
\end{aligned}$$

First we prove that the G_i's are representable by smooth A-algebras. In fact we consider the functor G_1': (A'-algebras) $\to$ (sets) defined by

$$G_1'(C) = \operatorname{Isom}(H' \otimes_{A'} C, H' \otimes_{A'} C)$$

as $G \otimes_A C$-spaces over C, for every A'-algebra C; we have $G_1 = \prod_{A'/A} G_1'$, i.e. G_1 is the Weil restriction of G_1' from A' to A (see [1, I, 1.6.6]). Hence by [1, I, 4.4.8] in order to prove that G_1 is representable by a smooth A-algebra it is enough to prove that G_1' is representable by a smooth A'-algebra.

Since H' is split by a faithfully flat A'-algebra S, the extended functor $G_1' \otimes_{A'} S$: (S-algebras) $\to$ (sets) is isomorphic to $G \otimes_A S$: in fact for every S-algebra D we have

$$(G_1' \otimes_{A'} S)(D) = \mathrm{Isom}_{G\otimes_A D\text{-spaces over } D}(H' \otimes_{A'} D, H' \otimes_{A'} D)$$
$$\simeq \mathrm{Isom}_{G\otimes_A D\text{-spaces over } D}(G \otimes_A D, G \otimes_A D) = (G \otimes_A S)(D).$$

Since $G \otimes_A S$ is representable by a smooth S-algebra, by [**8**, I, Proposition 1], G_1' is representable by a smooth A'-algebra. In the same way we prove that G_2, G_3 are representable by smooth A-algebras.

Now we consider the functor K: (A-algebras) $\to$ (sets) defined as the kernel

$$K(B) = \mathrm{Ker}(G_2(B) \underset{\theta_1 \cdot \theta_3}{\overset{\theta_2}{\rightrightarrows}} G_3(B))$$

for every A-algebra B, where θ_i are the maps induced by the homomorphisms ε_i: $A' \otimes_A A' \to A' \otimes_A A' \otimes_A A'$; by [**8**, I, Proposition 2], K is representable by an A-algebra of finite presentation.

If we prove that K is smooth, by [**5**, Theorem 1.8] we have that the map $K(A) \to K(\bar{A})$ is surjective, and hence we can lift any descent datum in $K(\bar{A})$ which corresponds to the $\bar{G}$-space $\bar{H}$ to a descent datum in $K(A)$ and obtain a G-space H over A which is a PHS over A by Proposition 2 (a) since it is split by a faithfully flat A-algebra of finite presentation $A \to A' \to S$. It remains to prove that K is smooth, i.e. for every A-algebra B and nilpotent ideal $I \subset B$ the map $K(B) \to K(B/I)$ is surjective. In fact let $\tilde{\alpha} \in K(B/I)$: $\tilde{\alpha}$ defines a $G \otimes_A B/I$-space $\tilde{H}$ over B/I which is a PHS since it is split by a faithfully flat B/I-algebra of finite presentation $B/I \to A' \otimes_A B/I \to S \otimes_A B/I$. From the bijection

$$H^1_{\mathrm{et}}(B, G \otimes_A B) \simeq H^1_{\mathrm{et}}(B/I, G \otimes_A B/I)$$

(see [**8**, I, Theorem 4 and II, Proposition 2]) it follows that $\tilde{H}$ can be lifted to a PHS over B. From this, using the fact that G_1 is smooth, we can see easily that there is an $\alpha \in K(B)$ whose image in $K(B/I)$ is $\tilde{\alpha}$.

As an application of Theorem 1 we deduce some properties of H-couples, some of them already known (see [**5** and **7**]).

COROLLARY 1. *Let $(A, \underline{a})$ be an H-couple, and*

(a) *let $P_n(A)$ be the set of isomorphism classes of projective A-modules with rank n, then $P_n(A) \simeq P_n(A/\underline{a})$; in particular* $\mathrm{Pic}(A) \simeq \mathrm{Pic}(A/\underline{a})$;

(b) *let $\mathrm{Az}_n(A)$ be the set of isomorphism classes of Azumaya A-algebras with rank n^2; then* $\mathrm{Az}_n(A) \simeq \mathrm{Az}_n(A/\underline{a})$;

(c) *let $\mathrm{Et}_n(A)$ be the set of isomorphism classes of étale finite A-algebras with rank n; then* $\mathrm{Et}_n(A) \simeq \mathrm{Et}_n(A/\underline{a})$;

(d) *let Q_{2n} be the set of isomorphism classes of quadratic A-modules of mark $(A^n \times A^n, q)$, $q(x, y) = x_1y_1 + \cdots + x_ny_n$ (see* [**1**, III, 5.2.3]); *then* $Q_{2n}(A) \simeq Q_{2n}(A/\underline{a})$.

PROOF. In fact all the sets considered above are of the form $H^1_{\mathrm{et}}(A, G)$ with G an affine smooth group scheme over A, precisely:

(a) $G = \mathrm{Gl}_n$, the general linear group of order n;

(b) $G = \mathrm{Pgl}_n$, the general projective linear group of order n;

(c) $G = S_n$, the symmetric group of order n;

(d) $G = D_{2n}$, the orthogonal group of order $2n$.

COROLLARY 2. *Let $(A, \underline{a})$ be an H-couple, with A-algebra over an algebraically closed field k, and let G be an affine algebraic group over k. Then*

$$H^1_{\mathrm{et}}(A, G \otimes_k A) \simeq H^1_{\mathrm{et}}(A/\underline{a}, G \otimes_k A/\underline{a}).$$

PROOF. In fact G is smooth.

COROLLARY 3. *Let $(A, \underline{a})$ be an H-couple and suppose that n is a unit in A. Let μ_n be the group of n-roots of unity, i.e. for every A-algebra B, $\mu_n(B) = \{b \in B \mid b^n = 1\}$. Then*

$$H^1_{\mathrm{et}}(A, \mu_n) \simeq H^1_{\mathrm{et}}(A/\underline{a}, \mu_n \otimes_A A/\underline{a}).$$

PROOF. In the above hypothesis μ_n is smooth.

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