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L^p-BOUNDEDNESS OF A CERTAIN CLASS OF MULTIPLIERS ASSOCIATED WITH CURVES ON THE PLANE. I

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ABSTRACT. L^p-unboundedness for $p \neq 2$ is proved in the case of multipliers which are constant along curves. In particular $y = x^n$ is included in the range of curves.

While the sharpest result on Bochner-Riesz multipliers remains open in $\mathbf{R}^n$, $n \geq 3$, it was obtained in the plane by L. Carleson and P. Sjölin [**1**] and later by C. Fefferman [**4**] through different methods. A. Cordoba [**2**] gets this result as a consequence of some estimates for multipliers supported on thin sets ("bump functions") and his result allows one to guess the "blow up" as we approach boundary results: the problem is to take Kakeya-Nikodym's set, which becomes "the witness" of estimates in $\mathbf{R}^2$.

This "geometric decomposition" method allows us to obtain other results. First of all it is possible to get Sjölin's theorems [**9**] on Bochner-Riesz multipliers associated to more general curves than the circumference (see [**8**]).

W. Littman, C. McCarthy and N. M. Riviere [**6**] obtained nonboundedness results for the "Schrödinger equation" multiplier $1/(\xi_0 - (\xi_1^2 + \cdots + \xi_n^2) + i)$, namely for $4 < p < \infty$. Nevertheless, Riviere [**7**] conjectured it should be bounded in some L^p with $p > 2$. The geometric method allows us to study a more general case; in the present paper we prove that the multiplier $\gamma(\xi_1 - \gamma(\xi_2))$ is only bounded in L^2, where $x = \gamma(y)$ is in a suitable class of curves in $\mathbf{R}^2$. The higher dimensional case follows from the present one after a theorem due to de Leeuw. The case $\gamma(x) = y^2$, which is just Riviere's, has been proved by C. Kenig and P. Tomas [**5**].

I would like to thank my teacher and friend Professor A. Cordoba for the orientation and advice that, in such a patient way, he has given to me concerning these problems.

We are going to refer to the class of curves $y = \gamma(x)$ such that:

(a) $y = \gamma(x)$ is $C^\infty(\mathbf{R})$ and $\gamma'(x) \to \infty$ as $x \to \infty$, and $\gamma''(x) > 0$ for large enough x.

(b) There exists a positive integer k_0 and constants L_γ and M_γ such that

$$1 < \frac{\gamma'(x_1)}{\gamma'(x_2)} < L_\gamma, \qquad 1 < \frac{\kappa(x_2)}{\kappa(x_1)} < M_\gamma$$

hold when x_1 and x_2 lie in $[2^k, 2^{k+1}]$ and $x_1 > x_2$ for every $k \geq k_0$; $\kappa(x)$ denotes the curvature of γ at x.

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(c) The length of the curve in the diadic interval $[2^k, 2^{k+1}]$ is larger than $\kappa(2^k)^{-1/2}$ when $k > k_0$.

The following curves satisfy the above hypotheses:

(1) $y = x^2$, $\alpha > 1$,

(2) $y = x(\log x)^\alpha$, $\alpha > 0$.

That is not the case of $\gamma(x) = e^x$, which does not satisfy (b). Consider $\phi \in L^p \cap L^\infty(\mathbf{R})$ for some $p \geqslant 1$, ϕ a positive function increasing on $\mathbf{R}^-$ and decreasing on $\mathbf{R}^+$; take

$$m(\xi_1, \xi_2) = \phi(\xi_2 - \gamma(\xi_1)), \qquad \xi = (\xi_1, \xi_2) \in \mathbf{R}^2,$$

and define

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi) \quad \text{for every } f \in S(\mathbf{R}^2).$$

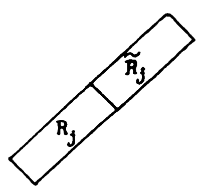
THEOREM I. *T is a bounded operator $L^p \to L^p$ iff $p = 2$.*

In order to prove Theorem I we need the following lemmas.

LEMMA 1 (KAKEYA'S SET). *Let α, μ, τ be positive numbers. Then there exists a $\delta_0 > 0$ such that given $\delta < \delta_0$ there is a set $E \subset \mathbf{R}^2$ and a family of pairwise disjoint rectangles $\{R_j\}$ whose directions are in the interval $[-\alpha/2, \alpha/2]$ and such that*

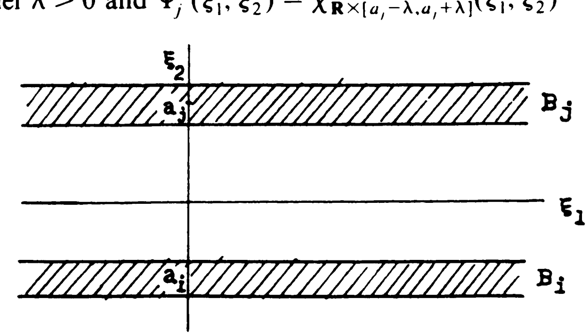
(i) *their dimensions are $\delta^{-1}/\mu \times \delta^{-1/2}\tau$;*

(ii) $|E \cap \tilde{R}_j| \geqslant \min(1/20, \mu\tau/20\alpha)|R_j|$, *where $\tilde{R}_j$ is the usual adjacent rectangle to R_j;*



(iii) $|E| \leqslant (8\mu\tau/\alpha)|\log\delta|^{-1}|\log|\log\delta||\cdot|\bigcup R_j|$.

Let us consider $\lambda > 0$ and $\Psi_j^\lambda(\xi_1, \xi_2) = \chi_{\mathbf{R}\times[a_j-\lambda, a_j+\lambda]}(\xi_1, \xi_2)$



where $\{a_j\}$ is a sequence of real numbers. Given $\{f_j\} \in L^p(l^2)$ we can state

LEMMA 2. *Assume $\|Tf\|_p \leqslant C_p\|f\|_p$ for every $f \in L^p$, then $\|(\sum|T_j^\lambda f_j|^2)^{1/2}\|_p \leqslant C \cdot C_p\|(\sum|f_j|^2)^{1/2}\|_p$ with c an absolute constant, where*

$$\left(T_j^\lambda f\right)\hat{}(\xi) = m(\xi)\Psi_j^\lambda(\xi)\hat{f}(\xi), \qquad \xi = (\xi_1, \xi_2) \in \mathbf{R}^2.$$

Its proof is immediate.

PROOF OF THEOREM I. We can suppose, without loss of generality, that $\phi \in L^1 \cap L^\infty(\mathbf{R})$. If $\|Tf\|_p \leqslant A\|f\|_p$, then $\|Sf\|_p \leqslant A\|f\|_p$ with the same constant where

$$\widehat{Sf}(\xi) = m(\xi)\chi_{[a,b]\times[\gamma(a),\gamma(b)]}(\xi_1, \xi_2)\hat{f}(\xi).$$

These a and b are chosen with the following properties:

(i) $a = 2^\rho$, $a > b$; $b < 2^{\rho+1}$, where ρ is a positive integer to be fixed later on.

(ii) The length L of the arc of the curve from a to b is $\kappa_1^{-1/2}$, where κ_1 is the curvature of γ at a.

(iii) Let us consider a $\zeta > 0$, to be fixed later on, only depending on ϕ and γ, and take ρ such that $\zeta \cdot \cos\alpha_1 < 1$, where α_1 is the angle between the tangent at a and the ox-axis.

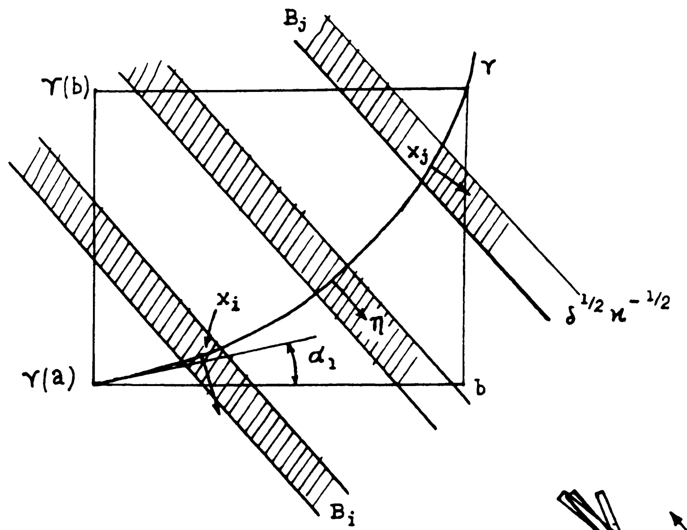
(iv) $\alpha_1 > \pi/4$.

(v) $1 < \kappa_1/\kappa_2 < M_\gamma$ where κ_2 is the curvature at b.

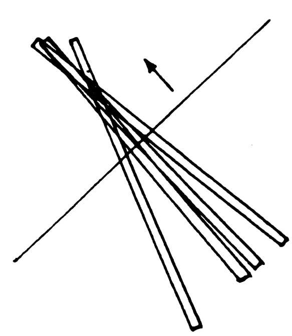
Consider the operator in Lemma 2 given by the strips whose direction is the normal to γ at the middle point of $[a, b]$, and $\lambda = \delta^{1/2}\kappa_1^{-1/2}$ (the a_j's will be fixed later on). There denote strips by B_j,

$$\left\|\left(\sum |T_j^\lambda f_j|^2\right)^{1/2}\right\|_p \leqslant A\left\|\left(\sum |f_j|^2\right)^{1/2}\right\|_p.$$

We want to find some functions $\{f_j\}$ which make A nonbounded. Let us choose the parameters in Lemma 1, $\delta = \cos\alpha_1$, $\mu = 8\zeta$, $\tau = \kappa_1^{1/2}/8$ and $\alpha = L\kappa_2/2$,



rotated Kakeya Set



Let R_j and E be the suitably rotated rectangles and set in Lemma 1. Now we choose $\{a_j\}$ such that the normal to the curve at x_j is the direction of R_j, where x_j is the point on γ whose projection is a_j. Since we chose $\alpha = L\kappa_2/2$ we are sure every x_j lies in the taken arc of the curve.

Let us remember the properties of $\{R_j\}$ and E,

(i) the dimensions of R_j are $\delta^{-1}/8\zeta \times \delta^{-1/2}\kappa_1^{1/2}/8$;

(ii) $|E \cap \tilde{R}_j| \geqslant |R_j|/20$;

(iii) $|E| \leqslant 16\zeta M_\gamma(\log|\log\delta|/|\log\delta|)|\cup R_j|$.

Let us take $f_j(x) = e^{ix_j\cdot x}\chi_{R_j}$. We claim that $|T_j^\lambda f_j(x)| \geqslant c(\gamma, \phi)$ for every $x \in \tilde{R}_j$,

$$|T_j^\lambda f_j(x)| = \left|\int_{\xi\in R} e^{-i\xi y}\phi(\xi_2 - \gamma(\xi_1))\chi_{B_j}(\xi)e^{ix_j(x-y)}\chi_{R_j}(x-y)\,d\xi\,dy\right|$$
$$\geqslant \left|\int_{\xi\in R}\phi(\xi_2 - \gamma(\xi_1))\chi_{B_j}(\xi)\int_{y\in x-R_j}\cos\langle x_j - \xi, y\rangle\,dy\,d\xi\right|$$

where $R = [a, b] \times [\gamma(a), \gamma(b)]$.

For $\xi \in R$ we have $\xi = (\xi_1, \xi_2) = \Gamma(t) + s\eta(\Gamma(t))$ where $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is $\gamma(t)$ parametrized by arclength and s is the normal distance.

Then

$$|T_j^\lambda f_j(x)| \geqslant \left|\int_{\substack{\xi\in R\\ s\in[-\zeta\delta,\zeta\delta]}}\phi(\xi_2 - \gamma(\xi_1))\chi_{B_j}(\xi)\int_{y\in x-R_j}\cos\langle x_j - \xi, y\rangle\,dy\,d\xi\right| - \int_{\substack{\xi\in R\\ s\notin[-\zeta\delta,\zeta\delta]}}\phi(\xi_2 - \gamma(\xi_1))\chi_{B_j}(\xi)|R_j|\,d\xi. \tag{1.1}$$

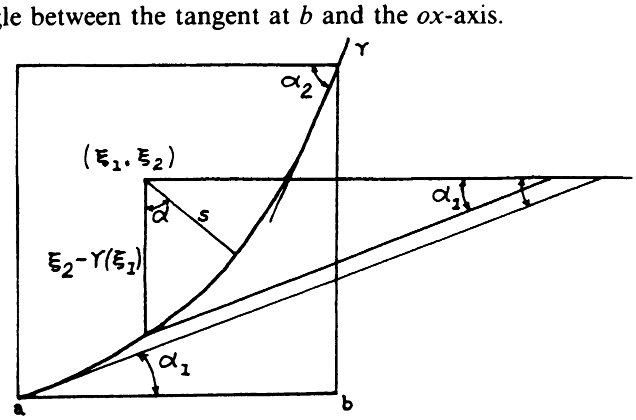
Let us look at the properties of ϕ and γ. In the case $s > 0$,

$$\xi_2 - \gamma(\xi_1) \leqslant \frac{s}{\cos\alpha} \leqslant \frac{s}{\cos\alpha_2},$$
$$\xi_2 - \gamma(\xi_1) \geqslant \frac{s}{2\cos\alpha_1}$$

and for $s < 0$,

$$\gamma(\xi_1) - \xi_2 \leqslant \frac{-2s}{\cos\alpha_2}, \qquad \gamma(\xi_1) - \xi_2 \geqslant \frac{-s}{2\cos\alpha_1}$$

where α_2 is the angle between the tangent at b and the ox-axis.



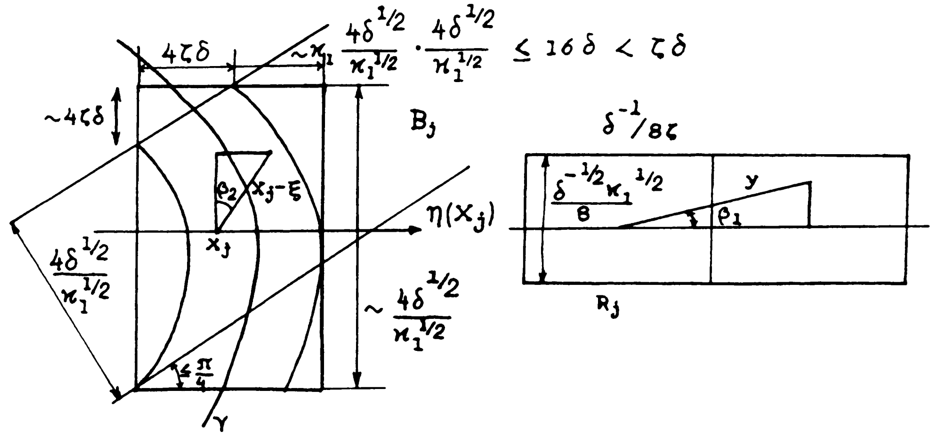
Therefore, when $s > 0$ we have

$$\phi(\xi_2 - \gamma(\xi_1)) \geqslant \phi\left(\frac{s}{\cos\alpha_2}\right),$$

$$\phi(\xi_2 - \gamma(\xi_1)) \leqslant \phi\left(\frac{s}{2\cos\alpha_1}\right),$$

and similar relationships when $s < 0$.

Let us estimate $\cos\langle x_j - \xi, y\rangle$ when $y \in x - \dot{R}_j$, $x \in \tilde{R}_j$ and $s \in [-\zeta\delta, \zeta\delta]$. $\chi_{[-\zeta\delta,\zeta\delta]}(s)\phi(\xi_2 - \gamma(\xi_1))\chi_{B_j}(\xi)$ is supported in a rectangle whose dimensions are $4\zeta \times 4\delta^{1/2}/\kappa^{1/2}$ since the curve is closer than $\zeta\delta$ to the tangent and the error due to the slope of the strip B_j is at most $4\zeta\delta < \delta^{1/2}/\kappa_1^{1/2}$ when δ is sufficiently small.



If

$$\begin{aligned}\langle x_j - \xi, y\rangle &= |x_j - \xi||y|\cos\left(\frac{\pi}{2} - \beta_1 - \beta_2\right)\\ &\leqslant |x_j - \xi||y||\sin\beta_1\cos\beta_2 + \cos\beta_1\sin\beta_2|\\ &\leqslant 2\zeta\delta|y|\cos\beta_1 + \frac{4\delta^{1/2}}{\kappa_1^{1/2}}\cdot\frac{\sigma^{-1/2}\kappa_1^{1/2}}{8} < 1,\end{aligned}$$

then $\cos\langle x_j - \xi, y\rangle > 1/4$ and therefore, we can drop down the absolute value in (1.1) and consider two integrals, one in the case $s > 0$ and the other when $s < 0$. Both integrals can be bounded in a similar way so we only consider the case $s > 0$.

(1.1) is bounded below by

$$\begin{aligned}&\left\{\frac{1}{4}\int_{s\in[0,\zeta\delta]}\phi\left(\frac{s}{\cos\alpha_2}\right)\chi_{B_j}(\xi)\,d\xi - \int_{s>\zeta\delta}\phi\left(\frac{s}{2\cos\alpha_1}\right)\chi_{B_j}(\xi)\,d\xi\right\}|R_j|\\ &\qquad\geqslant |R_j|\left\{\frac{1}{4}\int_{s\in[0,\zeta\delta]}\phi\left(\frac{s}{\cos\alpha_2}\right)\chi_{B_j}(\Psi(s,t))(1+s\kappa(t))\,ds\,dt\right.\\ &\qquad\qquad\left.-\int_{s>\zeta\delta}\phi\left(\frac{s}{2\cos\alpha_1}\right)\chi_{B_j}(\Psi(s,t))(1+s\kappa(t))\,ds\,dt\right\};\end{aligned}$$

after changing

$$\xi = \Psi(s, t) \geqslant \frac{1}{64\zeta}\left\{\frac{1}{8L_\gamma}\int_{u\in[0,\zeta L_\gamma]}\phi(u)\,du - 16\int_{u>\zeta/2}\phi(u)\,du\right\}$$

by taking $\nu = L_\gamma 16^2/(L_\gamma 16^2 + 1)$, $0 < \nu < 1$, and ζ such that

$$\min\left\{\int_{u\in[0,\zeta L_\gamma]}\phi(u)\,du, \int_{u\in[0,\zeta/2]}\phi(u)\,du\right\} \geqslant \nu\int_0^\infty \phi(u)\,du.$$

Then

$$|T_j^\lambda f_j(x)| \geqslant \frac{1}{64\zeta}\,\frac{\|\phi\|_1}{26L_\gamma} = c(\gamma, \phi) \quad \text{if } x \in \tilde{R}_j.$$

We apply the methods in **[4]** and obtain

$$A > cM_\gamma^{-(p-2)/2}\zeta^{-(p-2)/2p}\left|\frac{|\log\sigma|}{\log|\log\delta|}\right|^{(p-2)/2}.$$

Since $\delta = \cos\alpha_1$, it suffices to choose $\rho \to \infty$ to get $A \to \infty$.

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