

NILPOTENCY OF HOMOTOPY PUSHOUTS

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ABSTRACT. Precise and effectively verifiable conditions are obtained for the pushout of a nilpotent space along a cofibration inducing a surjection of fundamental groups to be nilpotent. As an application we show that homotopical localization preserves cofiber sequences of nilpotent spaces.

1. Introduction. It is well known that simply-connected spaces are closed under homotopy pushouts, but not under homotopy pullbacks. (For homotopy pullbacks and pushouts see [1, 5 and 6].) Closing the class of simply-connected spaces with respect to homotopy pullbacks gives the class of those spaces all of whose path-components are nilpotent. Here we investigate how far nilpotent spaces are stable under homotopy pushouts.

Since nilpotency indicates relations, while homotopy pushouts have considerable “freeness” in homotopy groups (cf. van Kampen’s Theorem), the answer is expected to be no. We give a precise answer in Proposition 3.2, for the case of a pushout of a nilpotent space along a cofibration inducing a surjection of fundamental groups. If the other two spaces involved are also nilpotent, we have Theorem 2.1, involving only integral homology and fundamental groups.

This result shows the paucity of homotopy colimits among nilpotent spaces. This shows that “one-sided homotopy model structures” (see [1]) occur even in homotopy theory. This also suggests that one of the reasons for the utility of nilpotent spaces is the generalized Zeeman comparison theorem (see [4]). It thus would be of interest to establish the broadest possible setting for comparison theorems. In fact we use a comparison theorem, via Proposition 3.4, where nilpotency enters only via *relative* homology groups.

We do not consider the case when g , in the notation of the next section, does not induce a surjection of the fundamental groups. Indeed in that case, we do not know whether the fundamental group of the pushout can be nilpotent.

A special case of Theorem 2.1 was proved in my doctoral thesis, written under the direction of Professor Peter Hilton. I thank him for helpful discussions. A Heller suggested, through Hilton, that I consider general pushouts; and I thank R. Sharpe for proving Lemma 4.3, which, sharpening my nebulous conjecture, tied up the loose ends.

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2. Statement of the result. We assume the following for the rest of the paper: There is a homotopy pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ Z & \xrightarrow{f'} & W \end{array}$$

where Y is nilpotent, X and Z have the homotopy type of connected CW-complexes and g_* from $\pi_1 X$ to $\pi_1 Z$ is surjective.

THEOREM 2.1. *Assume further that either Z is simply connected or, both X and Z are nilpotent. Then W is nilpotent if and only if one of the following holds:*

- (1) $f_*: \pi_1 X \rightarrow \pi_1 Y$ is surjective.
- (2) $H_*(g)$, the reduced homology of the mapping cone of g , is trivial.
- (3) There is a prime p such that $[\pi_1 W: f'_*(\pi_1 Z)]$ is a power of p and each $H_i(g)$ is annihilated by some power of p .

This is a special case of Proposition 3.2. The later is proved in the next two sections.

We now make an application of the theorem above to settle a question in [3, p.77].

COROLLARY 2.2. *Let $X \xrightarrow{f} Y \rightarrow W$ be a cofiber sequence of nilpotent spaces and P a collection of primes. Then $X_P \rightarrow Y_P \rightarrow W_P$ is also a cofiber sequence.*

We have the hypothesis of Theorem 1 fulfilled, with Z a point. So either (1) f induces a surjection of fundamental groups or (2) $H_i(g) = \tilde{H}_{i-1}(X)$ is trivial for all i or (3) there is a prime p such that $\pi_1 W$ is a finite p -group and each $\tilde{H}_i(X)$ is annihilated by some power of p .

By the case proved in [3, p. 77], and the argument used, we need consider only cases (2) and (3), and only show that C , the cofiber of f_p , is nilpotent. Note that $\pi_1 C = \text{coker}(f_p)_* = (\text{coker } f_*)_p = (\pi_1 W)_p$.

In case (2), X is homotopy equivalent to a point. Hence so is X_p and thus C is Y_p .

In case (3), we distinguish two subcases: (a) p is inverted (i.e. p is not in P) or (b) p is not inverted. In (a), $(\pi_1 W)_p$ is trivial, i.e. C is simply connected. In case (b), X is P -local, i.e. X_p is X . Also, $\pi_1 W$ is P -local i.e. $\pi_1 C$ is $\pi_1 W$. By Theorem 2.1, C is nilpotent.

3. Proof of Theorem 2.1. Without loss of generality (using results of [1]), we can assume that W is a CW-complex, Y and Z are subcomplexes such that $Y \cup Z$ is W and $Y \cap Z$ is X .

Now $\pi_1 W = \pi$ is nilpotent: π is the pushout of $\pi_1 Y$ along $g_*: \pi_1 X \rightarrow \pi_1 Z$ which is surjective so $\pi_1 W$ is a quotient of $\pi_1 Y$.

By II.2.18, and II.2.19 (pp. 70–71) of [3], W will be nilpotent iff $\pi_1 W$ is nilpotent and acts nilpotently on the homology of \tilde{W} , the universal cover of W .

Let \tilde{Y}, \tilde{Z} and \tilde{X} be the restriction of \tilde{W} to Y, Z and X respectively. Then $\tilde{Y} \cup \tilde{Z}$ is \tilde{W} and $\tilde{Y} \cap \tilde{Z}$ is \tilde{X} . Since $\pi_1 W$ is a quotient of $\pi_1 Y$, \tilde{Y} is connected. Now we need

LEMMA 3.1. *Let Y be nilpotent and \tilde{Y} a connected cover of Y with G as the group of covering transformations. Then G acts nilpotently on the homology of \tilde{Y} .*

PROOF. First we reduce to the case where \tilde{Y} is a regular cover of Y . Let \bar{Y} be the cover of Y corresponding to the normalizer of the image of $\pi_1 \tilde{Y}$ in $\pi_1 Y$. Then \tilde{Y} is a regular cover of \bar{Y} , \bar{Y} is nilpotent and \tilde{Y} has the same group of covering transformations over \bar{Y} and Y .

So let \tilde{Y} be a regular cover of Y . Then \tilde{Y} is the homotopy fiber of the obvious map from Y to $K(G, 1)$. The action of G on \tilde{Y} as by covering transformations and by $\pi_1 K(G, 1)$ are freely homotopic. Now apply Corollary 2.2 (p. 71) of [2].

Now we return to the proof of Theorem 2.1: Since π acts nilpotently on each $H_i(\tilde{Y})$, it acts nilpotently on each $H_i(\tilde{W})$ iff it acts nilpotently on each $H_i(\tilde{W}, \tilde{Y})$. By excision, $H_i(\tilde{W}, \tilde{Y})$ is $H_i(\tilde{Z}, \tilde{X})$.

Let π^1 be $f_*^!(\pi_1 Z)$ and \bar{Z} be the cover of Z corresponding to $\ker f_*^!$. Since g_* is surjective, \bar{X} , the restriction of \bar{Z} to X , is connected. π^1 acts via covering transformations on the pair (\bar{Z}, \bar{X}) . I claim that (\bar{Z}, \bar{X}) is homeomorphic as π -pair to $\pi \times_{\pi^1} (\bar{Z}, \bar{X})$: We have the following squares, in which the vertical maps are principal bundles; and which are equivalent to pullbacks.

$$\begin{array}{ccccccc}
 \tilde{W} & \text{---} & E\pi & \bar{Z} & \text{---} & E\pi & \pi \times_{\pi^1} E\pi & \text{---} & E\pi \\
 \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow b & & \text{pr} \downarrow \\
 W & \text{---} & E\pi/\pi & Z & \text{---} & E\pi/\pi^1 & E\pi/\pi^1 & \text{---} & E\pi/\pi
 \end{array}$$

$E\pi$ is a contractible CW-complex on which π acts freely and cellularly. Here all maps except a, b are the obvious ones, $b[g, x] = [x]$, $a[g, x] = gx$. The compositions $Z \rightarrow W \rightarrow E\pi/\pi$ and $Z \rightarrow E\pi/\pi^1 \rightarrow E\pi/\pi$ are homotopic: they induce the same map on the fundamental group level and $E\pi/\pi$ is a $K(\pi, 1)$. This proves the claim.

Let C_* be the cellular chain complex functor. By the above remark $C_*(\bar{Z}, \bar{X})$ is isomorphic to $\mathbf{Z}\pi \otimes_{\pi^1} C_*(\bar{Z}, \bar{X})$. Since $\mathbf{Z}\pi$ is free over $\mathbf{Z}\pi^1$, $H_*(\bar{Z}, \bar{X})$ is $\mathbf{Z}\pi \otimes_{\pi^1} H_*(\bar{Z}, \bar{X})$. We can apply Theorem 4.1 (next section) to decide when π acts nilpotently on each $\mathbf{Z}\pi \otimes_{\pi^1} H_i(\bar{Z}, \bar{X})$. The conditions are:

- (a) π^1 act nilpotently on each $H_i(\bar{Z}, \bar{X})$.
- (b) Either (1), (2) or (3) below holds:
 - (1) $\pi = \pi^1$.
 - (2) $H_*(\bar{Z}, \bar{X}) = 0$.

(3) Let N be the largest normal subgroup of π contained in π^1 . Then there is a prime p such that $[\pi : N]$ is a power of p and each $H_i(\bar{Z}, \bar{X})$ is annihilated by a power of p .

We recast the above result. First $\pi = \pi^1$ iff $f_*(\pi_1 X) = \pi_1 Y$: Clearly the latter implies the former. Let $\pi = \pi^1$, then $H_1(W, Z) = H_1(Y, X)$ is trivial; so $f_*: H_1 X \rightarrow H_1 Y$ is surjective. Since $\pi_1 Y$ is nilpotent, this implies that f_* is surjective.

Next $[\pi : N]$ is a power of p iff $[\pi : \pi^1]$ is: Again the “only if” part is trivial. Let $[\pi : \pi^1]$ be a power of p . We can find a chain $\pi^1 \subset \pi^2 \subset \dots \subset \pi^r = \pi$ where each π^i is normal in π^{i+1} of index a power of p . So $\pi^1 \rightarrow \pi$ and $\pi^1/N \rightarrow \pi/N$ are isomorphisms away from p . Since π/π^1 is finite, so is π/N . Thus there is l , a power of p s.t. $(\pi/N)^l \subset \pi^1/N$. Since $(\pi/N)^l$ is normal in π/N , $(\pi/N)^l$ is contained in N/N . By Cauchy’s theorem, the order of π/N is a power of p .

Finally, we can apply Proposition 3.4 below with \mathcal{C} , the quasi-ideal of π -modules annihilated by a power of p , to conclude

PROPOSITION 3.2. *In the above notation, W is nilpotent if and only if (a) and (b) are true:*

- (a) π^1 acts nilpotently on $H_i(\bar{Z}, \bar{X})$ for all i .
- (b) Either f_* maps $\pi_1 X$ onto $\pi_1 Y$ or $H_*(g)$ is trivial or there is a prime p such that $[\pi : \pi^1]$ is a power of p and each $H_i(g)$ is annihilated by a power of p .

To state Proposition 3.4, we need the notion of a quasi-ideal of G -modules, G being a group. Let \mathcal{C} be a class of (left) G -modules containing the zero module. Then \mathcal{C} is a quasi-ideal (1) if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of G -modules, B is in \mathcal{C} iff both A and C are; (2) if A is in \mathcal{C} and B is a nilpotent right G -module, $\text{Tor}_i^G(B, A)$ is in \mathcal{C} for all i . (Note that Tor_i is the tensor product.)

Then we have the following

LEMMA 3.3. *Let G act nilpotently on A . Then A is in \mathcal{C} iff $\mathbf{Z} \otimes_G A$ is.*

PROOF. Let H be the semidirect product of A and G . As is well known, we have epimorphisms

$$H_{ab}^{\otimes i} \otimes (A/\Gamma_H A) \rightarrow \Gamma_H^i A / \Gamma_H^{i+1} A.$$

The tensor product on the left can be over \mathbf{Z} or $\mathbf{Z}G$: they are equal. We also have the identifications $A/\Gamma_H A = A/\Gamma_G A = \mathbf{Z} \otimes_G A$. Hence we see that if $A/\Gamma_H A$ is in \mathcal{C} , so are $\Gamma_G^i A / \Gamma_G^{i+1} A$ and hence by downward induction on i , $\Gamma_G^i A$ is in \mathcal{C} for all i . The converse is trivial.

PROPOSITION 3.4. *Let C_* be a positive chain complex of flat G -modules. Suppose further that G acts nilpotently on $H_i(C_*)$ for all i . Let \mathcal{C} be a quasi-ideal of G -modules. Then $H_i(C_*)$, $0 \leq i \leq q$, are in \mathcal{C} iff $H_i(\mathbf{Z} \otimes_G C_*)$, $0 \leq i \leq q$, are in \mathcal{C} .*

PROOF. We have the universal coefficient spectral sequence $E_{pq}^2 = \text{Tor}_p^G(\mathbf{Z}, H_q(C_*)) \Rightarrow H_{p+q}(\mathbf{Z} \otimes_G C_*)$. This gives the only if part.

Let $H_i(\mathbf{Z} \otimes_G C_*)$ be in \mathcal{C} for $0 \leq i \leq q$. Note that $H_0(\mathbf{Z} \otimes_G C_*)$ is $\mathbf{Z} \otimes_G H_0(C_*)$. This, by Lemma 3.3 shows that $H_0(C_*)$ is in \mathcal{C} . We will show that $H_s(C_*)$ is in \mathcal{C} by induction on s .

Let $H_i(C_*)$ be in \mathcal{C} for $i < s$. Then E_{pi}^2 is in \mathcal{C} for $i < s$. So E_{pi}^r is in \mathcal{C} for $i < s$. Since $H_s(\mathbf{Z} \otimes_G C_*)$ and $E_{s-i,i}^\infty$ are in \mathcal{C} for $i < s$, $E_{0,s}^\infty$ is in \mathcal{C} . By descending induction on r , we can easily show that $E_{0,s}^r$ are in \mathcal{C} for all $r \geq 2$. Since $E_{0,s}^2$ is $\mathbf{Z} \otimes_G H_s(C_*)$, the induction is completed by applying Lemma 3.3.

4. Nilpotency of extended modules. Throughout this section G is a group, H a subgroup of G , not equal to G and A a nonzero H -module.

Let N be the largest normal subgroup of G contained in H . If X is a set on which G acts and R is any ring, RX is the free R -module with X as a basis, and we let G act on RX via its action on X . If A is an H -module, G acts on $\mathbf{Z}G \otimes_H A$ by the right translation.

THEOREM 4.1. G acts nilpotently on $\mathbf{Z}G \otimes_H A$ if and only if H acts nilpotently on A and there is a prime p such that G/N is a finite p -group and A is annihilated by a power of p .

PROOF. We will use without explicit mention the fact that $\mathbf{Z}G$ and $\mathbf{Z}G/\mathbf{Z}H$ are free over $\mathbf{Z}H$.

Since $A = \mathbf{Z}H \otimes_H A$ is a submodule of $\mathbf{Z}G \otimes_H A$, H acts nilpotently on A if G acts nilpotently on $\mathbf{Z}G \otimes_H A$. We assume that H acts nilpotently on A . We will prove that the rest of the condition is necessary and sufficient for G to act nilpotently on $\mathbf{Z}G \otimes_H A$.

Now G acts nilpotently on $\mathbf{Z}G \otimes_H A$ iff it acts nilpotently on $\mathbf{Z}G \otimes_H (\Gamma_H^i A / \Gamma_H^{i+1} A)$ for all $i \geq 0$ ($\Gamma_H^i A$ is trivial for large i). Each $\Gamma_H^i A / \Gamma_H^{i+1} A$ is annihilated by a power of p iff the same is true of A . So we can assume that H acts trivially on A . Let X be G/H . Then $\mathbf{Z}G \otimes_H A$ is $\mathbf{Z}X \otimes_{\mathbf{Z}} A$.

From now on all tensor products will be over \mathbf{Z} .

We need

LEMMA 4.2. Let K be a field of characteristic p . Then G acts nilpotently on KX iff p is positive and g/n is a finite p -group.

PROOF OF LEMMA 4.2. Let G act nilpotently on KX . I claim that X is finite: Let $\sum a(x)x$ be a nonzero element of KX fixed by G . Then $a(xg) = a(x)$ for all $g \in G$, $x \in X$. Since G acts transitively on X , $a(x)$ is nonzero for all x in X . Hence X is finite, so G/N is finite.

Let X be finite. Then by the Engel-Kolchin Theorem [7, pp. 5, 79] G acts nilpotently on KX iff for all g in G , the eigenvalues of g acting on KX are all 1.

Let g be in G , x in X and g^l fix x . Let λ be any l th root of unity. For computing eigenvalues, we can pass to any extension of K , in particular to $K(\lambda)$. Over $K(\lambda)$, λ is an eigenvalue of g with eigenvector $\sum_{i=0}^{l-1} \lambda^{-i} (xg^i)$. If μ is an eigenvalue of g , and g^l is in N , μ^l is an eigenvalue of g^l and so μ^l is 1. Thus G acts nilpotently on KX iff g^l is in N , and $\lambda^l = 1$ implies λ is 1.

Let q be any prime dividing the order of G/N . So there is g in G , not in N such that g^q is in N . The primitive q th root of unity will be different from 1 iff char K is not q . This proves Lemma 4.2.

Returning to the proof of Theorem 4.1, we prove the "if" part. Let $p^i A$ be trivial. Then $\mathbf{Z}X \otimes (p^i A / p^{i+1} A)$ is isomorphic to $(\mathbf{Z}/p)X \otimes (p^i A / p^{i+1} A)$. By Lemma 2, G acts nilpotently on the latter if G/N is a finite p -group.

We now prove the “only if” part. First we will show that for some prime p A is p -torsion: Let a be a nonzero element of A . If a is torsion-free, $\mathbf{Z}X = \mathbf{Z}X \otimes \mathbf{Z}a$ is a sub- G -module of $\mathbf{Z}X \otimes A$. Hence G acts nilpotently on $\mathbf{Z}X \otimes \mathbf{Q} = \mathbf{Q}X$. This contradicts Lemma 4.2. Let p be a prime dividing the order of a . Then we can find b in A of order p . Then $(\mathbf{Z}/p)X = \mathbf{Z}X \otimes \mathbf{Z}b$ is a sub- G -module of $\mathbf{Z}X \otimes A$. By Lemma 4.2, G/N is a finite p -group. Since G/N can be a finite p -group for at most one prime p , it follows that for some prime p , G/N is a finite p -group and A is p -torsion.

Let a be a nonzero element of A . Let the order of a be p^s . Let the nilpotency index for the action of G on $\mathbf{Z}X \otimes A$ be n . Then we will show that s is bounded above by a number depending only on n and p . This will prove Theorem 4.1.

Since G/N is a finite p -group, we can find g in G s.t. g is not in H but g^p is: Now G/N , being a finite p -group, is nilpotent. So H/N is subnormal in G/N . Choose $K \supset H$ s.t. H is normal in K and g in k s.t. GH has index p in K/H .

Let C_p be the cyclic group of order p , with a generator τ . Let Y be $\{H, gH, \dots, g^{p-1}H\}$. Then C_p acts on $\mathbf{Z}Y \otimes \mathbf{Z}a$ via $\tau(g^iH \otimes ta) = g^{i+1}H \otimes ta$. Clearly $\Gamma_{C_p}^i(\mathbf{Z}Y \otimes \mathbf{Z}a)$ is contained in $\Gamma_G^i(\mathbf{Z}X \otimes A)$. So C_p acts nilpotently on $\mathbf{Z}Y \otimes \mathbf{Z}a = (\mathbf{Z}/p^s)C_p$ with index at most n . The next lemma shows that s is at most $(n - 1)/(p - 1)$, completing the proof.

LEMMA 4.3 (R. SHARPE). *The nilpotency index for the C_p -action on $(\mathbf{Z}/p^s)C_p$ is $s(p - 1) + 1$.*

PROOF. The integer m is an upper bound for the nilpotency index if

$$(1 - \tau)^m (\mathbf{Z}/p^s)C_p$$

is trivial, i.e. if $(1 - \tau)^m$ is in $p^s \mathbf{Z}C_p$. Let ζ be a primitive p th root of unity over \mathbf{Q} . Now $(1 - \tau)^m$ is in $p^s \mathbf{Z}C_p$ if we can find polynomials f, g in the indeterminate s.t. $p^s f(\lambda) = (1 - \lambda)^m + (1 - \lambda^p)g(\lambda)$. Putting $\lambda = 1$, we see that $f(1) = 0$. So $f(\lambda) = (1 - \lambda)f_1(\lambda)$. So $(1 - \tau)^m$ is in $p^s \mathbf{Z}C_p$ iff there exist f_1 and g s.t.

$$p^s f_1(\lambda) = (1 - \lambda)^{m-1} + [(1 - \lambda^p)/(1 - \lambda)]g(\lambda).$$

Since $(1 - \lambda^p)/(1 - \lambda)$ is the minimal polynomial of ζ , this is true iff p^s divides $(1 - \zeta)^{m-1}$ in $\mathbf{Z}[\zeta]$. In $\mathbf{Z}[\zeta]$, $1 - \zeta$ is a prime and p is of the form unit $(1 - \zeta)^{p-1}$. (See [8, p. 262].) Hence p^s divides $(1 - \zeta)^{n-1}$ iff $n - 1 \geq s(p - 1)$. This completes the proof.

REMARK. The basic idea of this lemma is due to R. Sharpe. In the preprint of this paper, the nilpotency index was given as $s(p - 1)$. I thank P. Hilton for pointing out the correct value.

Addendum. After this paper was completed, my attention was drawn to two papers of R. H. Lewis. They appear as [9 and 10] in the references. In [9] he proves, in addition to a Blakers-Massey Theorem: Let X be nilpotent and $A \subset X$ be a cofibration. Then X/A is nilpotent if and only if $\mathbf{Z}\pi_1 X \otimes H_i A$ are all nilpotent $\pi_1 X$ -modules. In [10] he proves, among other things, the special case of our Theorem 4.1 in which H is trivial and G and A are finitely generated. In the topological part,

both the proofs are quite similar. However in the algebraic part, our Lemmas 4.2 and 4.3 enable us to deal with the nonfinitely generated case. Lewis' proof seems not to generalize easily. Furthermore our Proposition 3.4 allows us to tackle pushouts and not just cofibers.

Since writing this paper, I have been informed that Gencalves Daciderg (unpublished) has obtained similar results.

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