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COLLECTIONWISE NORMALITY IN SCREENABLE SPACES

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ABSTRACT. We show that if there is any normal screenable space which is not paracompact, there is one which is not collectionwise normal.

We use *space* to mean a Hausdorff topological space.

If a normal Moore space fails to be paracompact (and metric) it does so because it it not collectionwise normal [**1**]. If a normal screenable space fails to be paracompact, it does so because it is not countably paracompact (or countably metacompact) [**2**]. A recent (consistency) example of a normal screenable nonparacompact space is collectionwise normal [**3**], and a natural observation of Tall is that normal screenable spaces are collectionwise normal with respect to paracompact sets. Thus the following question arose: are all normal screenable spaces collectionwise normal? The purpose of this note is to prove that the answer is no, assuming the existence of any normal screenable nonparacompact space X. Imitating the construction of "Bing's G" [**4**] we construct a normal screenable space consisting of some isolated points together with a closed discrete family of ω_1 copies of X which cannot be contained in disjoint open sets. We thus construct a machine which, for each X, constructs a G^* as follows:

I. Assume that X is a normal screenable nonparacompact space.

Let $S = \{s\colon X \to [0,1] \mid s \text{ is continuous}\}$.

Let $\mathcal{A} = \{A\colon \omega_1 \to S\}$.

Let $G = \{g\colon \mathcal{A} \to [0,1]\}$.

For all $\alpha \in \omega_1$ and $x \in X$, let $f_{\alpha x}$ be the term of G defined by $f_{\alpha x}(A) = A(\alpha)(x)$; let $F_\alpha = \{f_{\alpha x} \mid \alpha \in \omega_1, x \in X\}$ and $F = \bigcup\{F_\alpha \mid \alpha \in \omega_1\}$.

A subbasic open set in the product topology on G is of the form $\{g \in G \mid g(A) \in B\}$ for some $A \in \mathcal{A}$ and open B in $[0,1]$.

We topologize G by using basic open sets from the product topology as basic open sets in G for points of F and declaring each point of $(G-F)$ to be isolated. This topology is certainly Hausdorff since it refines the product topology.

II. Since singletons in $G-F$ are open, F is closed. If $f_{\alpha x} \in U = \{g \in G \mid g(A) \in B\}$ for some $A \in \mathcal{A}$ and open B in $[0,1]$, then $A(\alpha)(x) \in B$ and, if $V = A(\alpha)^{-1}(B)$, then V is open in X and $\{f_{\alpha y} \mid y \in V\} \subset U$. Similarly, if $x \in V$, open in X, there is an $s \in S$ with $x \in s^{-1}([0,1)) \subset V$. Define $A \in \mathcal{A}$ by $A(\alpha) = s$ and $A(\beta)(x) = 1$ for

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all $\beta \neq \alpha$ and $x \in X$. Then $(\{g \in G \mid g(A) \in [0,1)\} \cap F) \subset \{f_{\alpha y} \mid y \in V\} \subset F_\alpha$. Thus the map from F_α to X defined by $f_{\alpha x} \to x$ is a homeomorphism. Also F_α is both open and closed in F and $\{F_\alpha \mid \alpha \in \omega_1\}$ is thus a closed discrete family in G.

LEMMA 1. *G is normal.*

PROOF. Suppose H and K are disjoint closed sets in G. We want disjoint open sets containing H and K, respectively. Since the points of $G - F$ are isolated we can assume that $(H \cup K) \subset F$. Since X is normal and the map $f_{\alpha x} \to x$ is a homeomorphism, for each $\alpha \in \omega_1$, we can choose a continuous $s_\alpha \colon X \to [0, 1]$ with $s_\alpha(\{x \in X \mid f_{\alpha x} \in H\}) = 0$ and $s_\alpha(\{x \in X \mid f_{\alpha x} \in K\}) = 1$. Define $A \in \mathcal{Q}$ by $A(\alpha) = s_\alpha$. Then $H \subset U = \{g \in G \mid g(A) \in [0, \frac{1}{2})\}$ and $K \subset V = \{g \in G \mid g(A) \in (\frac{1}{2}, 1]\}$. Since $U \cap V = \varnothing$ and U and V are open in G, the proof of Lemma 1 is complete.

III. Our desired screenable normal not collectionwise normal space G^* will be a closed subset of G.

Since X is normal, but not countably paracompact [5], there is an open cover $\{O_n \mid n \in \omega\}$ of X such that, if $D_n = O_n - \bigcup_{m<n} O_m$ and T_n is open and $T_n \supset D_n$, then $\{T_n \mid n \in \omega\}$ is not point finite.

If $n \in \omega$ and $\alpha \in \omega_1$, let $\mathcal{Q}_{\alpha n}$ be the set of all A in $\mathcal{Q}$ such that

(1) $A(\beta)(x) = 1$ for all $\beta \neq \alpha$ in ω_1 and $x \in X$, and

(2) $A(\alpha)^{-1}([0, 1))$ intersects D_n and is contained in O_n.

Let $\mathcal{Q}_\alpha = \bigcup \{\mathcal{Q}_{\alpha n} \mid n \in \omega\}$.

We let G^* be the set of all g in G such that

(1) if $n \in \omega$, there is at most one $\alpha \in \omega_1$ with $g(A) \in [0, 1)$ for some $A \in \mathcal{Q}_{\alpha n}$, and

(2) if $g(A)$ and $g(A')$ are in $[0, 1)$ for some A and A' in $\mathcal{Q}_\alpha$, then $A(\alpha)^{-1}([0, 1)) \cap A'(\alpha)^{-1}([0, 1)) \neq \varnothing$.

Observe that $F \subset G^*$. Since the points of $G - F$ are isolated, G^* is a closed (and hence normal) subset of G.

LEMMA 2. *G^* is screenable.*

PROOF. Suppose that U is an open cover of G^*. We want to construct a σ-disjoint refinement V of U covering F. Then $V \cup \{\{g\} \mid g \in G^* - F\}$ will be a σ-disjoint refinement of U covering G^*.

Fix $\alpha \in \omega_1$. For each $x \in X$, $f_{\alpha x} \in U_x \in \mathcal{U}$. Since $x \in D_n$ for some n, there is an $A_x \in \mathcal{Q}_{\alpha n}$ with $x \in T_x = \{y \in X \mid A_x(\alpha)(y) \in [0, 1)\} \subset \{y \in X \mid f_{\alpha y} \in U_x\}$. Since X is screenable there is a σ-disjoint refinement $\mathcal{W}$ of $\{T_x \mid x \in X\}$ covering X; say $\mathcal{W} = \bigcup \{\mathcal{W}_i \mid i \in \omega\}$ and the members of each $\mathcal{W}_i$ are disjoint. For $W \in \mathcal{W}$ choose $x_W \in X$ with $W \subset T_{x_W}$ and let $U_W = \{g \in U_x \mid g(A_{x_W}) \in [0, 1)\}$. For each $w \in W \in \mathcal{W}$, choose $A_{wW} \in \mathcal{Q}_\alpha$ such that $w \in A_{wW}(\alpha)^{-1}[0, 1) \subset W$. Let $V_{wW} = \{g \in G^* \mid g(A_{wW}) \in [0, 1)\}$. Then, for $W \in \mathcal{W}$, define $W^* = U_W \cap (\bigcup \{V_{wW} \mid w \in W\})$. Define $\mathcal{W}_{\alpha i n} = \{W \in \mathcal{W}_i \mid x_W \in D_n\}$ and $\mathcal{V}_{\alpha i n} = \{W^* \mid W \in \mathcal{W}_{\alpha i n}\}$.

Let $\mathcal{V}_{in} = \bigcup \{\mathcal{V}_{\alpha i n} \mid \alpha \in \omega_1\}$ and $\mathcal{V} = \bigcup \{\mathcal{V}_{in} \mid i \in \omega, n \in \omega\}$. Certainly $\mathcal{V}$ is an open cover of F refining $\mathcal{U}$. We claim that the members of each $\mathcal{V}_{in}$ are disjoint and thus $\mathcal{V}$ is σ-disjoint.

To see this, first suppose that for some α, W and W' are in $\mathcal{W}_{\alpha in}$. If $w \in W$ and $w' \in W'$, A_{wW} and $A_{w'W'}$ belong to $\mathcal{A}_\alpha$; and $A_{wW}(\alpha)^{-1}[0,1) \subset W$, $A_{w'W'}(\alpha)^{-1}[0,1) \subset W'$, and $W \cap W' = \varnothing$. So, by (2) in the definition of G^*, no $g \in G^*$ is in $(V_{wW} \cap V_{w'W'})$; hence $W^* \cap W'^* = \varnothing$.

Next, suppose $W \in \mathcal{W}_{\alpha in}$ and $W' \in \mathcal{W}_{\beta in}$ for some $\alpha \neq \beta$ in ω_1. Since $A_{x_W} \in \mathcal{A}_{\alpha n}$ and $A_{x_{W'}} \in \mathcal{A}_{\beta n}$ (the $A_{x_{W'}}$ being defined with β rather than α, fixed), by (1) of the definition of G^*, there is no $g \in G^*$ in $U_W \cap U_{W'}$. Thus $W^* \cap W'^* = \varnothing$; and our proof that the members of $\mathcal{V}_{in}$ are disjoint is complete.

LEMMA 3. *G^* is not collectionwise normal.*

PROOF. Suppose that $F_\alpha \subset U_\alpha$, open in G^*. We show that $\{U_\alpha \mid \alpha \in \omega_1\}$ are not disjoint.

Choose a countable basis $\mathcal{B}$ for $[0,1]$.

Fix $\alpha \in \omega_1$. If $x \in X$, there is a finite set P_x of pairs $\langle A, B\rangle$ with $A \in \mathcal{A}$ and $B \in \beta$ such that $f_{\alpha x} \in U_x = \{g \in G^* \mid g(A) \in B$ for all $\langle A, B\rangle \in P_x\} \subset U_\alpha$. Without loss of generality we assume for all $\langle A, B\rangle$ in P_x, $1 \notin B$ unless $A(\alpha)(x) = 1$. Let $T_x = \{y \in X \mid A(\alpha)(y) \in B$ for all $\langle A, B\rangle \in P_x\}$ and $M_x = \{m \in \omega \mid$ there is $\langle A, B\rangle \in P_x$ with $A \in \mathcal{A}_{\alpha m}$ and $B \subset [0,1)\}$. By our definition of D_n, there are x and y in X with $x \in T_y$ and $y \in D_n$ for some $n > (\sup M_x)$. Observe that since $y \in D_n$, if $m \in M_y$, then $m \geqslant n$; so $M_y \cap M_x = \varnothing$. Let x_α and y_α denote this chosen x and y for our fixed α and $P_{\alpha x}$, $P_{\alpha y}$, $M_{\alpha x}$, and $M_{\alpha y}$ the associated P_x, P_y, M_x and M_y.

By the standard Δ-system argument, one can find an uncountable subset Δ of ω_1 such that

(1) for all $\alpha \in \Delta$, all M_{x_α} are the same set which we call M_x and all M_{y_α}'s are the same set which we call M_y;

(2) there are i and j in ω such that, for all $\alpha \in \Delta$, the $P_{\alpha x}$'s all have i terms $\langle A_{\alpha 0}, B_{\alpha 0}\rangle, \ldots, \langle A_{\alpha(i-1)}, B_{\alpha(i-1)}\rangle$, and the $P_{\alpha y}$'s all have j terms $\langle A'_{\alpha 0}, B'_{\alpha 0}\rangle, \ldots, \langle A'_{\alpha(j-1)}, B'_{\alpha(j-1)}\rangle$;

(3) for $\alpha \in \Delta$, if $k < i$, all $B_{k\alpha}$'s are the same set, called B_k, and, if $l < j$, all B'_{lj}'s are the same set, called B'_l;

(4) there are subsets I of i and J of j such that, for all $\alpha \in \Delta$, if $k \in I$, all $A_{\alpha k}$'s are the same and, if $l \in J$, all $A'_{\alpha l}$'s are the same. And if $A = A_{\alpha k}$ for some $k \in i - I$ or $A = A'_{\alpha l}$ for some $l \in j - J$, then A is neither $A_{\beta h}$ for some $\beta \neq \alpha$ and $h < i$ nor $A_{\beta m}$ for some $\beta \neq \alpha$ and $m < j$.

Choose $\alpha \neq \beta$ in Δ. Then define $g \in G$ by

(1) $g(A) = 1$ if $A \notin \{A_{\alpha k} \mid k < i\} \cup \{A'_{\beta l} \mid l < j\}$;

(2) $g(A) = A(\alpha)(x_\alpha)$ if $A = A_{\alpha k}$ for some $k < i$;

(3) $g(A) = A(\beta)(y_\beta)$ if $A \in \{A'_{\beta l} \mid l < j\} - \{A_{\alpha k} \mid k < i\}$.

If we can show that $g \in U_\alpha \cap U_\beta$ our proof is complete. By (2), if $g \in G^*$, $g \in U_\alpha$. If for some $l < j$ and $k < i$, $A'_{\beta l} = A_{\alpha k}$, then $A_{\alpha k} = A_{\beta k}$ and $A'_{\beta l} = A'_{\alpha l}$ and, since $x_\alpha \in T_{y_\alpha}$, $A'_{\alpha l}(\alpha)(x_\alpha) \in B'_l$. Thus, since $g(A'_{\beta l}) = A_{\alpha k}(\alpha)(x_\alpha) = A'_{\alpha l}(\alpha)(x_\alpha)$, $g(A'_{\beta l}) \in B'_l$. This, together with (3), shows that, if $g \in G^*$, $g \in U_\beta$. So it remains to prove that, $g \in G^*$.

If $g(A) \in [0,1)$ for any $A \in \mathcal{A}_\delta$, then δ is α or β and $A \in \{A_{\alpha h} \mid h < i\} \cup \{A'_{\beta l} \mid l < j\}$.

If, for some $h < i$ and $m \in \omega$, $A_{\alpha h} \in \mathcal{A}_{\alpha m}$ and $g(A_{\alpha h}) \in [0,1)$, then $m \in M_x$. If $l < j$, $m \in \omega$, $A'_{\beta l} \in \mathcal{A}_{\beta m}$, and $g(A'_{\beta l}) \in [0,1)$, then, certainly, $m \in M_y$ unless $A'_{\beta l} = A_{\alpha k}$ for some $k < i$. In this case, as shown above, $g(A'_{\beta l}) \in B'_l$, and thus $m \in M_y$. Since $M_x \cap M_y = \varnothing$, g satisfies (1) of the definition of G^*.

If $g(A_{\alpha h}) \in [0,1)$ for some $h < i$, $x_\alpha \in A_{\alpha h}(\alpha)^{-1}[0,1)$; so (2) in the definition of G^* is satisfied for any A and A' in $\mathcal{A}_\alpha$.

If $g(A) \in [0,1)$ for some $A \in \mathcal{A}_\beta$, then A is A'_{kl} for some $l < j$. If $g(A) = A(\alpha)(y_\beta)$, then, since $x_\beta \in T_{y_\beta}$, $x_\beta \in A(\beta)^{-1}[0,1)$. Otherwise, $A = A_{\alpha k}$ for some $k < i$ and $g(A) = A(\alpha)(x_\alpha)$ and $A_{\alpha k} = A_{\beta k}$. Since $g(A) \neq 1$, $1 \notin B_k$, and $x_\beta \in A(\beta)^{-1}[0,1)$. So (2) in the definition of G^* is satisfied for all A and A' in $\mathcal{A}_\beta$ and the proof of our lemma is complete.

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