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CONCERNING EXACTLY $(n, 1)$ IMAGES OF CONTINUA

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ABSTRACT. A surjective mapping $f\colon X \to Y$ is exactly $(n, 1)$ if $f^{-1}(y)$ contains exactly n points for each $y \in Y$. We show that if Y is a continuum such that each nondegenerate subcontinuum of Y has an endpoint, and if $2 \leqslant n < \infty$, then there is no exactly $(n, 1)$ mapping from any continuum onto Y. However, if Y is a continuum which contains a nonunicoherent subcontinuum, then such an $(n, 1)$ mapping exists. Therefore, a Peano continuum is a dendrite if and only if for each n $(2 \leqslant n < \infty)$ there is no exactly $(n, 1)$ mapping from any continuum onto Y. We also show that for each positive integer n there is an exactly $(n, 1)$ mapping from the Hilbert cube onto itself.

1. Introduction. In the early 1940s a sequence of papers appeared which studied the existence of exactly $(n, 1)$ mappings defined on various classes of continua. (A mapping $f\colon X \to Y$ is *exactly* $(n, 1)$ if $f^{-1}(y)$ contains exactly n points, for each $y \in Y$.) It was shown by Harrold [5], Roberts [10] and Civin [3] that there is no exactly $(2, 1)$ mapping defined on a closed n-cell ($n = 1, 2, 3$), and the problem for $4 \leqslant n < \infty$ remains open. Other relevant papers are those of Harrold [6, 7], Gilbert [4], Martin and Roberts [8], Borsuk and Molski [1] and Mioduszewski [9].

A related problem is the following: Which continua are the *images* of some continuum under an exactly $(n, 1)$ mapping, where $2 \leqslant n < \infty$? Some partial answers were noted by Harrold [6] who showed that no arc has this property and that an exactly $(n, 1)$ image of a finite graph must contain a copy of S^1. We show that if Y is a continuum each of whose nondegenerate subcontinua has an endpoint and if $2 \leqslant n < \infty$, then there is no exactly $(n, 1)$ mapping from any continuum onto Y. We also show that there exist exactly $(n, 1)$ mappings from continua onto any non-hereditarily unicoherent continuum. Thus we can conclude, if Y is a Peano continuum and $2 \leqslant n < \infty$, that Y is a dendrite if and only if there is no $(n, 1)$ mapping from any continuum onto Y.

In particular, if $2 \leqslant n$, $m < \infty$, then an m-cell is the exactly $(n, 1)$ image of some continuum (compare [3, 5 and 10]). Actually, we are able to show that the continuum may be taken to be an AR, and this fact permits us to construct an exactly $(n, 1)$ mapping of the Hilbert cube onto itself.

2. The main results. A *continuum* is a compact connected Hausdorff space. An element e of continuum Y is an *endpoint* of Y if e admits arbitrarily small open

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neighborhoods with one-point boundary. A *cutpoint* of Y is an element p of Y such that $Y - \{p\}$ is not connected.

LEMMA. *Let Y be a continuum with an endpoint e and let $2 \leqslant n < \infty$. If there is an exactly $(n, 1)$ mapping f from a continuum X onto Y, then there is a proper subcontinuum Y_1 of Y such that $f^{-1}(Y_1)$ is connected.*

PROOF. Let $f^{-1}(e) = \{x_1, \ldots, x_n\}$ and let $U_1, U_2, \ldots, U_n$ be mutually disjoint open subsets of X such that $x_i \in U_i$ for each $i = 1, \ldots, n$, and let $U = \bigcup_{i=1}^{n}\{U_i\}$. Since $f^{-1}(e) \subset U$ and $X - U$ is compact, there is an open subset V of Y such that $e \in V$ and $f^{-1}(\bar{V}) \subset U$. Since e is an endpoint of Y, we may assume $\bar{V} - V = \{p\}$. Thus p is a cutpoint of Y and $Y_1 = Y - V$ is a proper subcontinuum of Y.

Suppose $f^{-1}(Y_1)$ is not connected; then $f^{-1}(Y_1) = A \cup B$ where A and B are disjoint nonempty closed subsets of X. Define subsets M and N of X as follows:

$$M = f^{-1}(\bar{V}) \cap \left[\bigcup \{U_i : f^{-1}(p) \cap A \cap U_i \neq \varnothing\}\right],$$

$$N = f^{-1}(\bar{V}) \cap \left[\bigcup \{U_i : f^{-1}(p) \cap A \cap U_i = \varnothing\}\right].$$

Clearly, M and N are disjoint. Moreover, since $f^{-1}(\bar{V}) \cap U_i = f^{-1}(\bar{V}) \cap \bar{U}_i$ for each i, each of the sets $f^{-1}(\bar{V}) \cap U_i$ is closed and hence M and N are closed. Since $A \subset f^{-1}(Y_1) = f^{-1}(Y - V)$ and $N \subset f^{-1}(\bar{V})$, it follows that $A \cap N \subset f^{-1}(p)$. By definition of N, $A \cap N \cap f^{-1}(p) = \varnothing$ and hence $A \cap N = \varnothing$. Since $B \subset f^{-1}(Y_1) = f^{-1}(Y - V)$ and $M \subset f^{-1}(\bar{V})$, it follows that $B \cap M \subset f^{-1}(p)$. Suppose there exists $q \in B \cap M$. Then$q \in f^{-1}(p)$ and, by definition of M, $q \in U_i$ where $f^{-1}(p) \cap A \cap U_1 \neq \varnothing$. Let $z \in f^{-1}(p) \cap A \cap U_i$. Then $\{q, z\} \subset f^{-1}(p) \cap U_i$ and, since $q \notin A$, q and z are distinct. Since $f^{-1}(p)$ contains only n points and since the sets U_1, $U_2, \ldots, U_n$ are mutually disjoint, there exists U_j such that $f^{-1}(p) \cap U_j = \varnothing$. Hence, $f^{-1}(\bar{V}) \cap U_j = f^{-1}(V) \cap U_j$ and thus $f^{-1}(\bar{V}) \cap U_j$ is both open and closed in X. Now $f^{-1}(\bar{V}) \cap U_j \neq X$ and $f^{-1}(\bar{V}) \cap U_j \neq \varnothing$ since it contains x_j as an element. This contradicts the hypothesis that X is connected and hence $B \cap \mathrm{M} = \varnothing$. However, $X(A \cup M) \cup (B \cup N)$, and hence $A \cup M$ and $B \cup N$ constiitute a separation of X, a contradiction. Therefore $f^{-1}(Y_1)$ is connected and the Lemma is proved.

THEOREM 1. *Let Y be a continuum such that every nondegenerate subcontinuum of Y has an endpoint. If $2 \leqslant n < \infty$, then there is no exactly $(n, 1)$ mapping from any continuum onto Y.*

PROOF. Suppose, on the other hand, there exists a continuum X and an exactly $(n, 1)$ mapping f of X onto Y. Consider the family $\{Y_\alpha\}$ of all subcontinua of Y such that each $f^{-1}(Y_\alpha)$ is connected, and let $\mathfrak{N}$ be a maximal nest chosen from this family. Let $Y_0 = \bigcap \mathfrak{N}$ and let $X_0 = f^{-1}(Y_0)$. Clearly, X_0 and Y_0 are continua and $f \mid X_0$ is exactly $(n, 1)$. In particular, Y_0 cannot be degenerate since X_0 is connected. But then Y_0 has an endpoint and so the Lemma contradicts the maximality of $\mathfrak{N}$.

Recall that a *dendrite* is a locally connected metrizable continuum which contains no simple closed curve. It is well known that every nondegenerate dendrite has an endpoint and that every subcontinuum of a dendrite is a dendrite [**11**]. Therefore, the following corollary is immmediate:

COROLLARY 1.1. *If $2 \leqslant n < \infty$, then there is no exactly $(n, 1)$ mapping from any continuum onto a dendrite.*

THEOREM 2. *If V is a continuum which contains a nonunicoherent subcontinuum and if $1 \leqslant n < \infty$, then there is an exactly $(n, 1)$ mapping from some continuum X onto Y.*

PROOF. The identity mapping $1: Y \to Y$ is $(1, 1)$ so we may assume $n > 1$. By hypothesis, there exist subcontinua A and B of Y such that $A \cap B$ is not connected; let P and Q be disjoint nonempty closed sets such that $A \cap B = P \cup Q$. For each $i = 1, \ldots, n-1$ let A_i, B_i and Y_i be distinct copies of A, B and Y. We adjoin these sets to Y as follows: each A_i is adjoined at P, each B_i is adjoined at Q and each Y_i is adjoined at $A \cup B$. The set $X = Y \cup [\bigcup_{i=1}^{n-1}\{A_i \cup B_i \cup Y_i\}]$ is a continuum in the adjunction topology and the natural mapping which identifies each Y_i with Y, each A_i with A and each B_i with B is exactly $(n, 1)$.

COROLLARY 2.1. *A Peano continuum Y is a dendrite if and only if for each n $(2 \leqslant n < \infty)$ there is no exactly $(n, 1)$ mapping from any continuum onto Y.*

PROOF. Since a dendrite is hereditarily unicoherent, this result is immediate from Corollary 1.1 and Theorem 2.

Theorem 1 can be used to show that certain continua other than dendrites cannot be exactly $(n, 1)$ images of any continuum $(2 \leqslant n < \infty)$. For example, the harmonic fan and the $\sin(1/x)$-continuum cannot be such images. It would be of interest to know if there is a tree-like continuum which is such an image.

The following corollary extends some of the above results to mappings which are *exactly n-component-to-one*, i.e., to mappings f such that $f^{-1}(Y)$ has exactly n components for each Y in the range of f.

COROLLARY 2.2. *The statements of Theorem* 1 *and Corollaries* 1.1 *and* 2.1 *remain valid for mappings which are exaclty n-component-to-one.*

PROOF. It suffices to show that if the continuum Y is the image of a continuum X under an exactly n-component-to-one mapping, then Y is the image of some continuum M under an exactly $(n, 1)$ mapping. If $f: X \to Y$ is the exactly n-component-to-one mapping, let $f = lm$ be the monotone-light factorization of f. That is, there is a continuum M, a monotone mapping $m: X \to M$, and a light mapping $l: M \to Y$ such that $f = lm$. It follows that l is exactly $(n, 1)$.

3. Exactly $(n, 1)$ mappings for the Hilbert cube. In the Introduction we noted that if $m \leqslant 3$ then there is no exactly $(2, 1)$ mapping defined on the m-cell, but that this problem is unsolved if $4 \leqslant m < \infty$. In this section we give a solution for $m = \aleph_0$.

THEOREM 3. *For each positive integer n there is an exactly $(n, 1)$ mapping from Q, the Hilbert cube, onto itself.*

PROOF. We assume $n > 1$ since the theorem is obvious if $n = 1$. Let I denote the line segment in Euclidean 3-space which joins $(0,0,0)$ and $(1,0,0)$. For each $i = 1, \ldots, n-1$ and each $j = 0, 1, 2, \ldots$ let $I_{i,j}$ denote the line segment joining $(2^{-j}, 0, 0)$ and $(2^{-j}(1 + (i-1)/n), 2^{-j}, 0)$ and let $D = I \cup \bigcup_{i,j}\{I_{i,j}\}$. The set D is a dendrite lying in the plane $z = 0$. Let $\Sigma(D)$ denote the suspension of D with vertices $(1,0,1)$ and $(1,0,-1)$ and let Y denote the 2-cell which is the intersection of $\Sigma(D)$ and the plane $y = 0$. Let $\sigma(Y)$ denote the bounding 1-sphere of Y, i.e., $\sigma(Y)$ is the suspension of the set $\{(0,0,0), (1,0,0)\}$. Let A be the line segment joining $(1,0,0)$ and $(2,0,0)$ and let $X = \Sigma(D) \cup A$. Note that X is an AR since the suspension of an AR is again an AR and the union of two ARs meeting in a single point is an AR.

There is a natural retraction $\pi\colon \Sigma(D) \to Y$ which has the property that $\pi^{-1}(y) = y$ if $y \in \sigma Y$, and $\pi^{-1}(y)$ consists of exactly n points if $y \in Y - \sigma(Y)$. (The map π folds each of the sets $\Sigma(I_{i,j})$ homeomorphically onto the suspension of the segment joining $(2^{-(j+1)},0,0)$ and $(2^{-j},0,0)$.) Define f from X onto Y by $f|\Sigma(D) = \pi$, $f|A$ *wraps the segment A around σY exactly $n - 1$ times*. It follows that f is an exactly $(n,1)$ mapping, and hence $f \times 1\colon X \times Q \to Y \times Q$ is exactly $(n,1)$ where 1 denotes the identity map on Q. By combining 44.1 and 22.1 of **[2]**, it follows that $X \times Q$ is a Hilbert cube. Clearly, $Y \times Q$ is a Hilbert cube. The theorem follows.

REMARK. In 2.7 of **[6]** it was shown that there does not exist an exactly $(n,1)$ mapping $1 < n < \infty$, from any continuum onto an arc. However, it follows from our proof of Theorem 3 that there do exist exactly $(n,1)$ mappings from continua onto k-cells for any $k \neq 1$: For $k = 2$, f is the desired mapping and, for $k > 2$, f crossed with the identity map on $[0,1]^{k-2}$ suffices. Moreover, for each k the domain of the mapping is a k-dimensional AR.

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