PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

CELL-LIKE DECOMPOSITIONS OF HOMOGENEOUS CONTINUA

JAMES T. ROGERS, JR.[1]

ABSTRACT. Certain decompositions of homogeneous continua are shown to be cell-like. In particular, the aposyndetic decomposition described by F. B. Jones of a homogeneous, decomposable continuum is cell-like, and we prove that any homogeneous decomposable continuum admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous, cell-like terminal continua so that the quotient space is an aposyndetic homogeneous continuum.

D. C. Wilson [8] has shown that a monotone, completely regular map $f\colon X \to Z$ of the n-dimensional continuum X onto the nondegenerate continuum Y has the property that $H^n(f^{-1}(z)) = 0$, for all z in Z. In particular, if $n = 1$, then the point inverses of f are acyclic continua.

More recently, Mason and Wilson [5] have shown that if $n = 1$, then the point inverses of f are tree-like continua, that is, f is a cell-like map.

Since the projection maps of a product space onto the factors are completely regular maps, one cannot, in general, extend the Mason-Wilson result to higher dimensions.

The impetus for the Mason-Wilson result, however, was the author's applications [6, 7] of completely regular maps to certain monotone decompositions of homogeneous continua. In this paper, we show that the completely regular, monotone maps arising as quotient maps of these decompositions are cell-like maps. In particular, Jones' aposyndetic decomposition of a homogeneous, decomposable continuum is a decomposition of that continuum into cell-like sets.

A continuum X is cell-like if each mapping of X into a compact ANR is inessential. If the map f is inessential, we write $f \simeq 0$. A continuum is cell-like if and only if it has trivial shape.

A map is cell-like if each of its point inverses is cell-like.

The following theorem is classical.

WIRECUTTING THEOREM. *Let A and B be closed subsets of the compact space M. If no connected subset of M intersects both A and B, then there exist disjoint closed subsets M_1 and M_2 of M such that $A \subset M_1$, $B \subset M_2$, and $M = M_1 \cup M_2$.*

A subcontinuum Z of the continuum X is said to be terminal if each subcontinuum Y of X such that $Y \cap Z \neq \varnothing$ satisfies either $Y \ddot{\subset} Z$ or $Z \subset Y$.

The proof of the next theorem is similar to that of [2, Theorem 2].

Received by the editors September 11, 1981.

1980 *Mathematics Subject Classification*. Primary 54F20, 54F50.

Key words and phrases. Cell-like decomposition, aposyndetic, homogeneous, completely regular map.

[1]This research was partially supported by NSF grant number MCS-8101565.

0002-9939/82/0000-0388/$02.00

THEOREM 1. *If A is a terminal subcontinuum of the continuum X, if B is a subcontinuum of X disjoint from A, and if $f\colon A \to Y$ is a map of A into the ANR Y, then there exists a map $F\colon X \to Y$ such that $F|A = f$ and $F|B \simeq 0$.*

PROOF. There exists an open set U in $X - B$ containing A and a map $g\colon U \to Y$ extending f. There exists an open neighborhood V of a point a in A such that $V \subset U$ and $g|V \simeq 0$, since Y is locally contractible. Since $A - V$ and $X - U$ are closed subsets of $X - V$ such that no connected subset of $X - V$ meets both $A - V$ and $X - U$, the Wirecutting Theorem implies that $X - V$ is the union of the disjoint closed sets X_1 and X_2, with $A - V \subset X_1$ and $X - U \subset X_2$.

Thus $X_1 \cup A$ and X_2 are disjoint closed subsets of X. Let $h\colon X \to I$ be a Urysohn map with $h(X_1 \cup A) = 0$ and $h(X_2) = 1$. Let $M = h^{-1}([0, \frac{1}{2}])$ and $N = h^{-1}([\frac{1}{2}, 1])$. Then $X = M \cup N$, $A \subset M$, $X - U \subset N$, and $M \cap N \subset V - A$. Since the map $g|M \cap N$ is homotopic to a constant map, there exists an inessential map $k\colon N \to Y$ such that $k|M \cap N = g|M \cap N$.

The map $F\colon X \to Y$ defined by

$$F(x) = \begin{cases} g(x), & x \in M, \\ k(x), & x \in N, \end{cases}$$

is the desired extension of f.

A map $g\colon X \to Z$ between continua is completely regular if for each $\delta > 0$ and each point z in Z, there exists an open set V in Z containing z such that if $z' \in V$ then there is a homeomorphism h from $g^{-1}(z)$ to $g^{-1}(z')$ such that $d(x, h(x)) < \delta$, for each x in $g^{-1}(z)$. Each completely regular map is open.

THEOREM 2. *Let $g\colon X \to Z$ be a monotone, completely regular map of the continuum X onto the nondegenerate continuum Z. Let z_1 be a point of Z. If $g^{-1}(z_1)$ is a terminal subcontinuum of X, then g is a cell-like map.*

PROOF. Let Y be a compact ANR, and let $f\colon g^{-1}(z_1) \to Y$ be a map. Let z_2 be another point of Z, and let $F\colon X \to Y$ be an extension of f such that $F|g^{-1}(z_2) \simeq 0$.

Since Y is a compact ANR, there exists $\epsilon > 0$ such that for any space W and for any two maps $\alpha, \beta\colon W \to Y$, $d(\alpha, \beta) < \epsilon$ implies $\alpha \simeq \beta$. Let δ be a positive number such that $d(x, x') < \delta$ implies $d(F(x), F(x')) < \epsilon$.

Let $Z_2 = \{z \in Z\colon F|g^{-1}(z) \simeq 0\}$. We show that Z_2 is open. Let $z \in Z_2$. Since g is completely regular, these exists an open set V in Z containing z such that if $z' \in V$, then there is a homeomorphism h from $g^{-1}(z)$ to $g^{-1}(z')$ such that $d(x, h(x)) < \delta$, for each x in $g^{-1}(z)$. Hence $d(F|g^{-1}(z'), F|g^{-1}(z) \circ h^{-1}) < \epsilon$, and so $F|g^{-1}(z') \simeq F|g^{-1}(z) \circ h^{-1}$. But the latter map is inessential, since $z \in Z_2$, and so $z' \in Z_2$. Hence Z_2 is open. A similar proof shows that Z_2 is closed.

Since Z_2 is a nonempty open and closed subset of the connected set Z, it follows that $Z_2 = Z$ and hence $f \simeq 0$. Thus each mapping of $g^{-1}(z_1)$ into a compact ANR is inessential, and hence $g^{-1}(z_1)$ is cell-like. Since point inverses of completely regular maps defined on continua are homeomorphic, this implies that g is a cell-like map.

The homeomorphism group H of a continuum X is said to respect the decomposition $\mathcal{G}$ of X if $G \in \mathcal{G}$ implies $h(G) \in \mathcal{G}$, for each homeomorphism h in H.

The following theorem is due to E. Dyer (see [**3**] for a simple proof).

THEOREM 3. *Let X and Y be nondegenerate metric continua and let $f\colon X \to Y$ be a monotone open surjection. Then there exists a dense G_δ-subset A of Y having* *the following property: for each $y \in A$, for each continuum $B \subset f^{-1}(y)$, for each x from the interior of B in $f^{-1}(y)$ and for each neighborhood U of B in X, there exists a continuum $Z \subset X$ containing B and a neighborhood V of y in Y such that $x \in Z^0$, $(f|Z)^{-1}(V) \subset U$ and $f|Z\colon Z \to Y$ is a monotone surjection.*

THEOREM 4. *Let X be a homogeneous continuum. Let $\mathcal{G}$ be a partition of X into terminal subcontinua such that the homeomorphism group of X respects $\mathcal{G}$. Then*

(1) *the partition $\mathcal{G}$ is a continuous decomposition of X.*
(2) *The quotient map $\pi\colon X \to Z$ of X onto the quotient space Z is completely regular.*
(3) *Z is a homogeneous continuum.*
(4) *The elements of $\mathcal{G}$ are mutually homeomorphic, indecomposable, homogeneous, cell-like continua.*

PROOF. Parts (1), (2), and (3) have been proved in [**7**, Theorem 4]. Since the homeomorphism group of X respects $\mathcal{G}$, it follows that the elements of $\mathcal{G}$ are mutually homeomorphic and homogeneous. Theorem 2 implies that the elements of $\mathcal{G}$ are cell-like. As in [**4**], Dyer's Theorem and the fact that the elements of $\mathcal{G}$ are terminal continua imply that the proper subcontinua of one (and hence all) of the elements of $\mathcal{G}$ have empty interiors in that element. Hence the elements of $\mathcal{G}$ are indecomposable.

An important application of this theorem is the following improvement of Jones' Aposyndetic Decomposition Theorem [**1**]. In the case that X is a curve, this improvement is already known from results of Rogers [**6**] and Mason and Wilson [**5**].

THEOREM 5. *Suppose X is a homogeneous, decomposable continuum. Then X admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous, cell-like terminal continua so that the quotient space is an aposyndetic homogeneous continuum.*

PROOF. Jones [**1**] has shown that his decomposition of X consists of terminal continua. Furthermore, the homeomorphism group of X respects this decomposition.

Question 1. Is each homogeneous, indecomposable, cell-like continuum tree-like?

Question 2. Can the aposyndetic decomposition of Theorem 5 raise dimension?

REFERENCES

1. F. B. Jones, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. **6** (1955), 735–740.
2. J. Krasinkiewicz, *Mapping properties of hereditarily indecomposable continua*, preprint.
3. ——, *On two theorems of Dyer*, preprint.
4. T. Maćkowiak and E. D. Tymchatyn, *Continuous mappings on continua*. II, preprint.
5. A. Mason and D. C. Wilson, *Monotone mappings on n-dimensional continua*, preprint.
6. J. T. Rogers, Jr., *Completely regular mappings and homogeneous, aposyndetic continua*, Canad. J. Math. **33** (1981), 450–453.
7. ——, *Decompositions of homogeneous continua*, Pacific J. Math. **99** (1982), 137–144.
8. D. C. Wilson, *Completely regular mappings and dimension*, Bull. Amer. Math. Soc. **76** (1970), 1057-1061.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118