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ON EXTREME POINTS OF SUBORDINATION FAMILIES

YUSUF ABU MUHANNA

ABSTRACT. Let F be the set of analytic functions in $U = \{z: |z| < 1\}$ subordinate to a univalent function f. Let $D = f(U)$. For $g(z) = f(\phi(z)) \in F$, let $\lambda(\theta)$ denote the distance between $g(e^{i\theta})$ and ∂D (boundary of D). We obtain the following results.

(1) If f' is Nevanlinna then $\int_0^{2\pi} \log \lambda(\theta)\, d\theta = -\infty$ if and only if

$$\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|)\, d\theta = -\infty.$$

(2) If g is an extreme point of the closed convex hull of F then

$$\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|)\, d\theta = -\infty,$$

for the case when D is a Jordan domain subset to a half-plane and f' is Nevanlinna.

1. Introduction. Let $U = \{z: |z| < 1\}$ and let A denote the set of functions analytic in U. Let B_0 denote the subset of A consisting of functions ϕ that satisfy $|\phi(z)| < 1$ for $z \in U$ and $\phi(0) = 0$.

Throughout this paper we assume that $f \in A$ and f is univalent in U. Let F denote the subset of A consisting of functions g that are subordinate to f in U. This means that $g \in A$, $g(0) = f(0)$ and $g(U) \subset f(U)$. These conditions are equivalent to the existence of $\phi \in B_0$ so that $g(z) = f(\phi(z))$. F is characterized by

$$g(z) = f(\phi(z)) \tag{1}$$

where $\phi \in B_0$. Equation (1) defines a one-to-one correspondence between F and B_0.

Let D denote $f(U)$. For $g \in F$, let

$$g(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta}). \tag{2}$$

Since $f \in H^p$, for $p < \frac{1}{2}$, $g(e^{i\theta})$ exists almost everywhere. Let $\lambda(\theta)$ denote the distance between $g(e^{i\theta})$ and ∂D where ∂D denotes the boundary of D. T. H. MacGregor and the author [1] proved that if f is convex, bounded, and if ∂D is sufficiently smooth, then g is an extreme point of F if and only if

$$\int_0^{2\pi} \log \lambda(\theta)\, d\theta = -\infty. \tag{3}$$

This result implies the well-known fact that ϕ is an extreme point of B_0 if and only if

$$\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|)\, d\theta = -\infty \tag{4}$$

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[2, p. 125]. Other results in this direction can be found in [1, 4, 5 and 7].

It was also proved [1] that when f is bounded, convex and ∂D is sufficiently smooth the correspondence between F and B_0 given by $g(z) = f(\phi(z))$ provides a one-to-one correspondence between the extreme points of B_0 and the extreme points of F. This is to say,

$$(5)\quad \int_0^{2\pi} \log \lambda(\theta)\, d\theta = -\infty \quad \text{if and only if} \quad \int_0^{2\pi} \log\big(1 - |\phi(e^{i\theta})|\big)\, d\theta = -\infty.$$

In §2, we prove that statement (5) holds for the case when f is univalent and f' is in the Nevanlinna class of analytic functions.

In §3, we give a necessary condition on the extreme points of the closed convex hull of F for the case when $D = f(U)$ lies in a half-plane and ∂D is a Jordan curve.

2. Functions subordinate to a univalent function with a Nevanlinna derivative. We let $d(z, \Gamma)$ denote the distance between z and a closed set Γ, $m(A)$ denote the Lebesgue measure of A and $\log^+ x = \max\{0, \log x\}$.

THEOREM 1. *Let f be analytic and univalent in U. Assume that f' is in the Nevanlinna class. Let D denote $f(U)$ and let F denote the set of functions subordinate to f. For $g(z) = f(\phi(z)) \in F$, let $\lambda(\theta)$ denote $d(g(e^{i\theta}), \partial D)$. Then*:

(a) $\int_0^{2\pi} \log^+ \lambda(\theta)\, d\theta$ *is convergent.*

(b) $\int_0^{2\pi} \log \lambda(\theta)\, d\theta = -\infty$ *if and only if* $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|)\, d\theta = -\infty$.

PROOF. Since f is univalent, it follows that

$$(6)\quad \frac{1}{4}\big(1 - |z|^2\big)|f'(z)| \le d(f(z), \partial D) \le \big(1 - |z|^2\big)|f'(z)|, \qquad z \in U$$

[8, p. 22]. When $g(e^{i\theta})$ and $\phi(e^{i\theta})$ exist and $|\phi(e^{i\theta})| < 1$, we obtain

$$(7)\quad \frac{1}{4}\big(1 - |\phi(e^{i\theta})|^2\big)|f'(\phi(e^{i\theta}))| \le \lambda(\theta) \le \big(1 - |\phi(e^{i\theta})|^2\big)|f'(\phi(e^{i\theta}))|.$$

Hence (7) implies that $\lambda(\theta) \le |f'(\phi(e^{i\theta}))|$ and consequently $0 \le \log^+ \lambda(\theta) \le \log^+ |f'(\phi(e^{i\theta}))|$. Since f' is Nevanlinna and ϕ is bounded, it follows that $f'(\phi(z))$ is also Nevanlinna. Hence $\log|f'(\phi(e^{i\theta}))| \in L^1$ and in particular $\log^+|f'(\phi(e^{i\theta}))| \in L^1$ [2, p. 16]. Therefore, $\int_0^{2\pi} \log^+ \lambda(\theta)\, d\theta$ is convergent, which is part (a).

Next, let $A = \{\theta\colon g(e^{i\theta})$ exists and $\lambda(\theta) = 0\}$. If $m(A) > 0$ then it follows that $\int_0^{2\pi} \log \lambda(\theta)\, d\theta = \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|)\, d\theta = -\infty$. Assume for the rest of the proof that $m(A) = 0$. This implies that (7) holds for almost all θ. The fact that $1 \le 1 + |\phi(e^{i\theta})| \le 2$ reduces (7) to

$$(8)\quad \frac{1}{4}\big(1 - |\phi(e^{i\theta})|\big)f'(\phi(e^{i\theta}))| \le \lambda(\theta) \le 2\big(1 - |\phi(e^{i\theta})|\big)|f'(\phi(e^{i\theta}))|$$

which also holds for almost every θ. Thus we conclude that $-\infty \le \int_0^{2\pi} \log \lambda(\theta)\, d\theta < M$, for some constant M, because $\log^+ \lambda(\theta) \in L^1$. This together with (8) implies that

(9)

$$-2\pi \log 4 + \int_0^{2\pi} \log|f'(\phi(e^{i\theta}))|\, d\theta + \int_0^{2\pi} \log\big(1 - |\phi(e^{i\theta})|\big)\, d\theta \le \int_0^{2\pi} \log \lambda(\theta)\, d\theta$$
$$\le \int_0^{2\pi} \log\big(1 - |\phi(e^{i\theta})|\big)\, d\theta + \int_0^{2\pi} \log|f'(\phi(e^{i\theta}))|\, d\theta + 2\pi \log 2$$

and so part (b) follows.

We close this section by noting that the conclusion of Theorem 1 is true for the case when g is subordinate to a close to convex function f. This is so because it was shown that $f' \in H^{1/3}$ [3] and thus f' is Nevanlinna.

3. Jordan domains. We let cA denote $\mathbf{C} \setminus A$.

LEMMA 1. *Let D be a bounded Jordan domain (∂D is a Jordan curve). Let g be a nonconstant bounded analytic function in U. If $g(e^{i\theta}) \in \bar{D}$ for almost all θ then $g(U) \subset D$.*

PROOF. Let $G = g(U)$. We want to show that $G \subset D$. We shall show first that $\partial\bar{G} \subset D$. Assume, to the contrary, that there is a point $w_0 \in \partial\bar{G}$ and $w_0 \notin \bar{D}$. Let $\varepsilon = d(w_0, D)$. Since $w_0 \in \partial\bar{G}$ and $w_0 \notin \bar{D}$, there exists a point $w_1 \in c\bar{G}$ and $w_1 \notin \bar{D}$ so that $|w_0 - w_1| < \varepsilon/2$. It follows that $d(w_1, D) > \varepsilon/2$. Let

$$h(z) = \frac{1}{g(z) - w_1}, \qquad z \in U, \tag{10}$$

$h(z)$ is analytic, bounded and $h(e^{i\theta}) = 1/(g(e^{i\theta}) - w_1)$ for almost all θ. Since $g(e^{i\theta}) \in \bar{D}$ for almost all θ, it follows that $|h(e^{i\theta})| \leq 2/\varepsilon$ for almost all θ. The Poisson Formula implies that $|h(z)| \leq 2/\varepsilon$ for every $z \in U$. This contradicts $|w_1 - w_0| < \varepsilon/2$. Hence $\partial\bar{G} \in \bar{D}$.

Next, we shall show that $L = \bar{G} \cap c\bar{D}$ is open. Let $w \in L$. Since $c\bar{D}$ is open, there is a neighborhood of w, $N_\varepsilon(w)$, so that $N_\varepsilon(w) \subset c\bar{D}$. $N_{\varepsilon/2}(w) \subset L$, because if not then $N_{\varepsilon/2}(w) \cap c\bar{G} \neq \varnothing$ and, since $N_{\varepsilon/2}(w) \cap \bar{G} \neq \varnothing$, one concludes that $N_{\varepsilon/2}(w) \cap \partial\bar{G} \neq \varnothing$. Thus there exists $w_0 \in N_{\varepsilon/2}(w) \cap \partial\bar{G}$ and $w_0 \notin \bar{D}$. This then contradicts the first part of the proof of the lemma. Hence L is open.

Let $u \in \partial L$ and assume that $u \notin \bar{D}$. Since $cL = c\bar{G} \cup \bar{D}$ it follows that every neighborhood of u, with radius less than $d(u, D)$, intersects $c\bar{G}$. This implies that $u \in \partial\bar{D}$ and consequently $u \in \bar{D}$. Hence $\partial L \subset \partial D$ and consequently $c\bar{D} = L \cup (c\bar{L} \cap c\bar{D})$. Since $c\bar{D}$ is connected (∂D is a Jordan curve) and since L is bounded, we conclude that L is empty and then $\bar{G} \subset \bar{D}$. This and Jordan's Theorem [7, p. 115] imply that $G \subset D$.

The following lemma is a generalization of Lemma 1.

LEMMA 2. *Let D be a Jordan domain subset to a half-plane H. Let g be a nonconstant function analytic in U so that $g(U) \subset H$. If $g(e^{i\theta}) \in \bar{D}$ for almost every θ then $g(U) \subset D$.*

PROOF. Let T be a möbius transformation that maps H onto U. Let $h(z) = T(g(z))$. $h(z)$ is bounded, analytic, $h(e^{i\theta}) = T(g(e^{i\theta}))$ exists for almost all θ and $h(e^{i\theta}) \in \overline{T(D)}$. Since T is a homeomorphism and ∂D is a Jordan curve, it follows that $\partial(T(D))$ is a Jordan curve. Hence, by Lemma 1, $h(U) \subset T(D)$ and consequently $g(U) \subset D$.

We now apply Lemma 2 to get the following theorem.

THEOREM 2. *Let f be a univalent analytic function in U. Assume that $D = f(U)$ is a Jordan domain subset to a half-plane H. Let F be the set of analytic functions subordinate to f. If g is an extreme point of the closed convex hull of F then $\int_0^{2\pi} \log \lambda(\theta)/(1 + \lambda(\theta))\, d\theta = -\infty$.*

REMARK. $\lambda(\theta)/(1+\lambda(\theta))$ can be replaced by $\lambda(\theta)$ when f is bounded.

PROOF. Assume that $\int_0^{2\pi}\log[\lambda(\theta)/(1+\lambda(\theta))]\,d\theta > -\infty$. Since $\lambda(\theta)/(1+\lambda(\theta)) < 1$, $\log[\lambda(\theta)/(1+\lambda(\theta))] \in L^1$. Let

$$h(z) = z\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{\lambda(t)}{1+\lambda(t)}\,dt\right\}. \tag{11}$$

It is known that $h \in H^\infty$ and $|h(e^{i\theta})| = \lambda(\theta)/(1+\lambda(\theta))$ for almost all θ [2, pp. 24, 126]. Since $|h(e^{i\theta})| \leqslant \lambda(\theta)$, it follows that $g(e^{i\theta}) \pm h(e^{i\theta}) \in \bar{D}$ for almost all θ. Moreover, $h \in H^\infty$ implies that $g(z) \pm h(z)$ is in a half-plane H_1, containing H, for all $z \in U$. Thus, by Lemma 2, it follows that $g(z) \pm h(z) \in D$ for all $z \in U$ and so $g(z) \pm h(z) \in F$. Since $h \not\equiv 0$, g cannot be an extreme point.

We come now to the main result of this section.

THEOREM 3. *Let f be a univalent analytic function in U. Assume that f' is Nevanlinna and $D = f(U)$ is a Jordan domain subset to a half-plane H. Let F be the set of analytic functions subordinate to f. If $g(z) = f(\phi(z))$ is an extreme point of the closed convex hull of F then $\int_0^{2\pi}\log(1-|\phi(e^{i\theta})|)\,d\theta = -\infty$.*

REMARK. In other words, $\{g \in F$: g is an extreme point of the closed convex hull of $F\} \subset \{f(\phi)$: ϕ is an extreme point of $B_0\}$.

PROOF. Theorem 2 implies that $\int_0^{2\pi}\log[\lambda(\theta)/(1+\lambda(\theta))]\,d\theta = -\infty$. Let $E = \{\theta$: $\lambda(\theta)$ exists and $\lambda(\theta) \leqslant 1\}$ and let $G = \{\theta$: $\lambda(\theta)$ exists and $\lambda(\theta) > 1\}$. $m(E \cup G) = 2\pi$. For $\theta \in E$, we have

$$\frac{\lambda(\theta)}{2} \leqslant \frac{\lambda(\theta)}{1+\lambda(\theta)} \leqslant \lambda(\theta) \tag{12}$$

and for $\theta \in G$, we have $1+\lambda(\theta) < 2\lambda(\theta)$ and so

$$\frac{1}{2} < \frac{\lambda(\theta)}{1+\lambda(\theta)} < 1. \tag{13}$$

(13) implies that $\int_G \log[\lambda(\theta)/(1+\lambda(\theta))]\,d\theta$ is convergent. Therefore,

$$\int_0^{2\pi}\log\frac{\lambda(\theta)}{1+\lambda(\theta)}\,d\theta = \int_E \log\frac{\lambda(\theta)}{1+\lambda(\theta)}\,d\theta = -\infty$$

and by (12) $\int_E \log\lambda(\theta)\,d\theta = -\infty$. Because of Theorem 1 this gives that $\int_0^{2\pi}\log\lambda(\theta)\,d\theta = -\infty$ and consequently $\int_0^{2\pi}\log(1-|\phi(e^{i\theta})|)\,d\theta = -\infty$.

REMARKS. 1. The conclusion of Theorem 3 follows for the case when f is convex. This is because $f(D)$ is a Jordan domain and $f' \in H^{1/2}$ [3].

2. Theorem 2 was proved by T. H. MacGregor and the author [1] for the case when f is convex and $f(U)$ is not a half-plane.

3. The converse of Theorem 3 does not hold in general. For example, the extreme points of F, when $f = (1+z)/(1-z)$, are characterized by

$$g = \frac{1+xz}{1-xz}, \qquad |x| = 1.$$

Other examples in [1 and 6] show this claim.

4. We conjecture that Theorems 1 and 3 hold for any unrestricted univalent function f.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF PETROLEUM AND MINERALS, DHAHRAN, SAUDI ARABIA