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ON INEQUALITIES OF PERIODIC FUNCTIONS AND THEIR DERIVATIVES

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ABSTRACT. The inequality $\|f\|_B \leq a_k \|f^{(k)}\|_B$ is proved for many spaces of periodic functions. An analogue for sequences is also given.

1. In his paper [**6**] in 1939 Northcott proved for periodic functions for which $\int_0^{2\pi} f(x)\,dx = 0$ and $f, \ldots, f^{(k-1)}$ are absolutely continuous that

$$\text{(1.1)}\quad \|f\|_\infty \leq a_k \|f^{(k)}(\cdot)\|_\infty \text{ where } a_k = \frac{4}{\pi}\sum_{i=1}^{\infty} i^{-k-1} \text{ for } k \text{ odd}$$

$$\text{and } a_k = \frac{4}{\pi}\sum_{i+1}^{\infty}(-1)^{i+1} i^{-k-1} \text{ for } k \text{ even}$$

and for each k, a_k is best possible and is actually achieved. Bellman [**2**] later showed that

$$\text{(1.2)}\qquad \|f(\cdot)\|_{2r} \leq a_k \|f^{(k)}(\cdot)\|_{2r}$$

holds. Here the constants a_k are the same as in (1.1), $r > 1$ is an integer, and f is any 2π-periodic function such that $\int_0^{2\pi} f(x)\,dx = 0$, $f^{(k-1)}$ is absolutely continuous, and $f^{(k)} \in L_{2r}$. The constants a_k in (1.2) are not best possible in general. For $r = 1$ the best constant is 1 by Wirtinger's inequality [**1**]. We will prove that if $L_{2r}(T)$ ($T \equiv [0, 2\pi]$ for $f(x)$ satisfying $f(0) = f(2\pi)$) is replaced by $L_p(T)$, $p \geq 1$, we still have

$$\text{(1.3)}\qquad \|f(\cdot)\|_p \leq a_k \|f^{(k)}(\cdot)\|_p$$

and for each k, a_k is best possible for L_1. We shall show that there is a whole class of Banach spaces for which an analogous result to (1.3) is valid.

In 1954 K. Fan, O. Taussky and J. Todd [**4**] proved a discrete analogue of the above that for a real finite sequence $x_1, \ldots, x_n$, $n \geq 2$, satisfying $\sum_{i=1}^n x_i = 0$

$$\text{(1.4)}\qquad \max|x_i| \leq \frac{n}{4}\max|x_{i+1} - x_i| \quad \text{for even } n \text{ and}$$

$$\max|x_i| \leq \frac{n^2-1}{4}\max|x_{i+1} - x_i| \quad \text{for odd } n$$

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and the (1.4) estimate is achieved and therefore is best possible. They proved also that for any positive integer r and defining $x_j \equiv x_i$ if $j = i \pmod n$,

$$(1.5)\qquad \left\{\sum_{i=1}^{n} |x_i|^r\right\}^{1/r} \geqslant \frac{n-1}{2}\left(\sum_{i=1}^{n} |x_{i+1} - x_i|^r\right)^{1/r} \quad \text{and}$$

$$\left\{\sum_{i=1}^{n} |x_i|^r\right\}^{1/r} \leqslant \frac{n^2-1}{12}\left(\sum_{i=1}^{n} |x_{i-1} - 2x_i + x_{i+1}|^r\right)^{1/r}$$

(where x_i may be complex). We shall prove for $1 \leqslant p < \infty$

$$(1.6)\qquad \left\{\sum_{i=1}^{n} |x_i|^p\right\}^{1/p} \leqslant b_n^k\left\{\sum_{i=1}^{n} |\Delta^k x_i|^p\right\}^{1/p}$$

where b_n is $n/4$ or $(n^2-1)/4$ for even or odd n respectively.

The constants are better than those in (1.5), p is not required to be an integer, and k can be any integer, not just 1 or 2.

2. The inequality on periodic functions. Using Northcott's result and a technique established in proving inequalities of Kolmogorov type, see [3], we have

THEOREM 2.1. *Suppose f is a 2π periodic function satisfying $\int_0^{2\pi} f(x)\,dx = 0$ and for which $f, \ldots, f^{(k-1)}$ are absolutely continuous and $f^{(k)} \in L_p(T)$ for some p, $1 \leqslant p < \infty$. Then*

$$(2.1)\qquad \|f\|_p \leqslant a_k \|f^{(k)}\|_p \quad \text{where } a_k \text{ is given by (1.1).}$$

PROOF. Let $F(x) = \int_0^{2\pi} f(x+t)g(t)\,dt$ where $g \in L_q$, $q^{-1} + p^{-1} = 1$, and $\|g\|_q = 1$. Obviously $F(x)$ satisfies $\int_0^{2\pi} F(x)\,dx = 0$ and $F^{(i)}(x)$ is absolutely continuous for $i < k$ and $F^{(k)}(x)$ is continuous. Therefore, $|F(0)| \leqslant \|F(x)\|_\infty \leqslant a_k \|F^{(k)}(\cdot)\|_\infty \leqslant a_k \|f^{(k)}\|_p$. Choosing g so that $|F(0)| = |\int_0^{2\pi} f(t)g(t)\,dt| = \|f\|_p$, we complete the proof.

In fact the following more general theorem can be proved equally easily. The theorem is applicable to Banach space B on which a group of isometries $S(t)$ is defined satisfying $S(2\pi) = S(0) = I$. We will say that $f \in B$ is in $D(A)$ if the strong limit of $(S(\eta) - I)f/\eta$ as $\eta \to 0+$ exists, and $f \in D(A^k)$ if $f \in D(A^{k-1})$ and $s\text{-}\lim_{\eta \to 0+} (S(\eta) - I)A^{k-1}f/\eta$ exists. We say $f \in D_w(A^k)$ if the limit exists in the weak sense and $f \in D_{w^*}(A^k)$ if the limit exists in the weak* sense (which is only possible if $B = X^*$).

THEOREM 2.2. *Let B be a Banach space, $S(t)$ a group of isometries and $f \in D(A^k)$ or $f \in D_w(A^k)$ or $f \in D_{w^*}(A^k)$, and suppose $\int_0^{2\pi} T(t) f\,dt = 0$ in the weak or weak* sense. Then*

$$(2.2)\qquad \|f\|_B \leqslant a_k \|A^k f\|_B \quad \text{where } a_k \text{ is given by (1.1).}$$

PROOF. For $f \in D(A^k)$ or $f \in D_w(A^k)$ choose $g \in B^*$ such that $\|g\|_{B^*} = 1$ and write $F(x) = \langle S(x)f, g\rangle$. Obviously $F(x), \ldots, F^{(k-1)}(x)$ are absolutely continuous, $F(x)$ periodic and $\int_0^{2\pi} F(x)\,dx = 0$, and therefore

$$|\langle f, g\rangle| = |F(0)| \leqslant \|F\|_\infty \leqslant a_k \|F^{(k)}(\cdot)\|_\infty \leqslant a_k \|A^k f\|.$$

Choosing g so that $|\langle f, g\rangle| \geq \|f\|_B - \varepsilon$, we complete the proof of this case. For $f \in D_{w^*}(A^k)$ we choose $g \in X$, $X^* = B$, and proceed similarly.

REMARK. There are many spaces satisfying the above, $L_p(T)$ Sobolev spaces, $L_p^r(T)$, Orlicz spaces on T and others. The weak* limit would be of interest for the space of functions of bounded variation or duals of Sobolev spaces.

THEOREM 2.3. *For $f \in L_1$ or $\alpha \in B.V.$ where $\int_0^{2\pi} f(x)\,dx = 0$ or $\int_0^{2\pi} d\alpha(x) = 0$ respectively we have $\|f\|_1 \leq a_k \|f^{(k)}\|_1$ or $\|\alpha\|_{B.V.} \leq a_k \|\alpha^{(k)}\|_{B.V.}$ or $\|f\|_1 \leq a_k \|f^{(k-1)}\|_{B.V.}$ where a_k is as defined in* (1.1) *and this a_k is best possible here.*

PROOF. The inequality was proved in Theorem 2.2. What is left is to show that a_k is best possible.

The extremum is achieved on a variant of the function given by Northcott. We will use notation and known results from a well-known paper of Kolmogorov [5]. Let

$$f_k(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x - (\pi/2)k)}{(2m+1)^{k+1}}.$$

Obviously $f_0(x) = f_k^{(k)}(x)$, $\|f_0\|_{\mathrm{B.V.}[T]} = 4$ and

$$\|f_k\|_{\mathrm{B.V.}[T]} = 4\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}}$$

for k odd and

$$\|f_k\|_{\mathrm{B.V.}[T]} = 4\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{k+1}}$$

for k even (see description of f_n [5, p. 236]). For $\alpha(x) = f_k(x)$ the equality is achieved. We can, therefore, approximate $f \in L_1$ by $f(x) = (1/h)\int_x^{x+h} d\alpha(t)$ ($\alpha(t) = f_k(t)$ here) and for small enough h the inequality is almost achieved. For $\alpha(t) \in \mathrm{B.V.}(T)$, $\|\alpha\|_{\mathrm{B.V.}} \leq a_k \|\alpha^{(k)}\|_{\mathrm{B.V.}}$ and $\|f\|_{L_1} \leq a_k \|f^{(k-1)}\|_{\mathrm{B.V.}}$ and a_k is best possible and the critical element is in the space. For $\|f\|_{L_1(T)} \leq a_k \|f^{(k)}\|_{L_1(T)}$, a_k is best possible and the critical element is not in the space.

3. The discrete case. In [4] the following theorem was proved.

THEOREM A. *If n real numbers $x_1, \ldots, x_n$ satisfy $\sum_{i=1}^n x_i = 0$, then*

$$\max |x_i| \leq b_n \max |x_{i+1} - x_i|$$

where $b_n = n/4$ for even n and $b_n = (n^2 - 1)/4n$ for odd n and b_n are best possible.

REMARK 3.1. We can replace real x_i by complex z_i. If $z_1, \ldots, z_n$ satisfies $\sum_{i=1}^n z_i = 0$, so do $z_i e^{i\theta}, \ldots, z_n e^{i\theta}$. We now choose θ such that $|z_l| = z_l e^{i\theta} = \max |z_i|$ (0 if there is more than one l possible, choose the first one) and let $x_i \equiv \operatorname{Re} z_i e^{i\theta}$. The sequence $x_1, \ldots, x_n$ satisfies the conditions of Theorem A and therefore,

$$\max |x_i| = |z_l| = \max |z_i| \leq b_n \max |\operatorname{Re}(z_{i+1} - z_i) e^{i\theta}| \leq b_n \max |z_{i+1} - z_i|.$$

We can also prove

THEOREM 3.2. *For $x_1, \ldots, x_n$, n complex numbers, $n \geqslant 2$, satisfying $\sum_{i=1}^n x_i = 0$ we have* $\max | x_i | \leqslant b_n^k \max | \Delta^k x_i |$ *and* $(\sum_{i=1}^n | x_i |^p)^{1/p} \leqslant b_n^k (\sum_{i=1}^n | \Delta^k x_i |^p)^{1/p}$ *for $p \geqslant 1$ where b_n is given in Theorem* A, $\Delta x_i \equiv x_{i+1} - x_i$, $\Delta^k x_i \equiv \Delta(\Delta^{k-1} x_i)$ *and $x_j \equiv x_i$ if $j = i \pmod n$.*

PROOF. The inequality $\max | x_i | \leqslant b_n^k \max | \Delta^k x_i |$ follows Theorem A and Remark 3.1 repreated k times since $\sum_{i=1}^n \Delta^l x_i = 0$. For any sequence $y_1, \ldots, y_n$ such that $(\sum_{i=1}^n | y_i |^q)^{1/q} = 1$, $p^{-1} + q^{-1} = 1$, or in case $p = 1$, $\max | y_i | = 1$ we can write $z_i = \sum_{j=1}^n x_{j+i} y_j$. Obviously $\sum z_i = \sum_{i=1}^n \sum_{j=1}^n x_{j+i} y_j = 0$ and $| z_i | \leqslant (\sum_{j=1}^n | x_j |^p)^{1/p}$ while $| \Delta^k z_i | \leqslant (\sum_{j=1}^n | \Delta^k x_j |^p)^{1/p}$.

We use the first part of the proof and write

$$| z_i | \leqslant b_n^k \max | \Delta^k z_i | \leqslant b_n^k \left(\sum_{j=1}^n | \Delta^k x_j |^p \right)^{1/p},$$

and choosing y_j appropriately $| z_i = (\sum_{j=1}^n | x_j |^p)^{1/p}$, our theorem follows.

One can prove in a similar way the following somewhat more general theorem.

THEOREM 3.3. *Let B be an n-dimensional Banach space of complex sequences, $x = (x_1, \ldots, x_n)$, such that $\sum_{i=1}^n x_i = 0$, $x_j \equiv x_i$, $j = i \pmod n$ and $\| \{x_{i+1}\} \| = \| \{x_i\} \|$. Then $\| \{x_i\} \|_B \leqslant b_n^k \| \{\Delta^k x_i\} \|_B$ where b_n is given in Theorem* A.

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