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APPROXIMATING THE ABSOLUTELY CONTINUOUS MEASURES INVARIANT UNDER GENERAL MAPS OF THE INTERVAL

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ABSTRACT. Let $\tau\colon I \to I$ be a nonsingular, piecewise continuous transformation which admits a unique absolutely continuous invariant measure μ with density function f^*. The main result establishes the fact that f^* can be approximated weakly by the density functions of a sequence of measures invariant under piecewise linear Markov maps $\{\tau_n\}$ which approach τ uniformly.

1. Introduction. Let τ be a nonsingular, measurable transformation from $I = [0, 1]$ into itself and let $\mathfrak{B}$ denote the Lebesgue measurable subsets of I. A measure μ defined on $(I, \mathfrak{B})$ is absolutely continuous if there exists a function $f\colon I \to [0, \infty)$, which is integrable with respect to Lebesgue measure m, i.e., $f \in \mathcal{L}_1(I, \mathfrak{B}, m) \equiv \mathcal{L}_1$, and for which

$$\mu(S) = \int_S f(x)\, m(dx) \quad \forall S \in \mathfrak{B}.$$

The measure μ is said to be invariant (under τ) if $\mu(\tau^{-1}S) = \mu(S)$ for all sets $S \in \mathfrak{B}$.

The Frobenius-Perron operator $P_\tau\colon \mathcal{L}_1 \to \mathcal{L}_1$ has proven to be a useful tool in the study of absolutely continuous invariant measures **[1, 2]**. It is defined by

$$(P_\tau f)(x) = \frac{d}{dx}\int_{\tau^{-1}[0, x]} f(s)\, m(ds).$$

The importance of P_τ lies in the fact that each of its fixed points is the density of a measure invariant under τ, i.e., if $P_\tau f^* = f^*$, then

$$\mu(\quad) = \int_{\cdot} f^*(x)\, m(dx)$$

is invariant under τ **[1]**.

In **[2]** a sequence of matrices $\{P_n\}$, depending on τ, is constructed and the following result obtained:

THEOREM 1. *Let $\tau\colon I \to I$ be a piecewise C^2 map with $\inf|\tau'| > 2$. If P_τ has a unique fixed point f^*, then the sequence $\{f_n\}$ of fixed points (regarded as functions on I) of $\{P_n\}$ converges to f^* in the $\mathcal{L}_1$-norm.*

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The proof of Theorem 1 depends on the fact that P_τ, where τ is expanding, reduces the variation of the function on which it acts [1]. The critical inequality is

$$\bigvee_0^1 P_\tau f \leq \alpha \| f \| + \beta \bigvee_0^1 f,$$

where V_0^1 denotes the variation on $[0,1]$ and $\| \quad \|$ the $\mathcal{L}_1$-norm. In the proof of Theorem 1, β must be less than 1 and this is truly only when $\inf|\tau'| > 2$.

When τ is nonexpanding, as for example if $\tau(x) = \gamma x(1-x)$, where γ can take on any value between 0 and 4, there are no known results similar to Theorem 1. The technique of [2] fails because $P_\tau f$ can have infinite variation for f of bounded variation.

In this note we shall obtain a result analogous to Theorem 1 for a large class of transformations τ which admit a unique absolutely continuous invariant measure. To do this we shall use the weak topology on the space of measures and the result will be of the form: $f_n \xrightarrow{\omega} f^*$, where ω denotes weak convergence. Although this may not appear to be a strong result, it is sufficient for most statistical purposes; for example, all the moments of the density f_n will be close to the corresponding ones for f^*, and

$$\mu_n(S) = \int_S f_n(x)\, m(dx) \to \int_S f^*(x)\, m(dx) \equiv \mu(S)$$

for any $S \in \mathfrak{B}$.

2. Notation. A piecewise continuous map τ_n: $I \to I$ is called Markov if there exist points $a_0 < a_1 < \cdots < a_{n-1} < a_n$ such that for $i = 0, 1, \ldots, n-1$, $\tau|_{I_i}$, where $I_i = (a_{i-1}, a_i)$, is a homeomorphism onto some interval $(a_{j(i)}, a_{k(i)})$. The partition $J_n = \{I_i^n\}_{i=1}^n$ is referred to as a Markov partition with respect to τ.

Now let τ: $I \to I$ be piecewise continuous and nonsingular. Partition $I \times I$ into an $n \times n$ grid and form the piecewise linear map τ_n by joining corner points of the grid in such a way that τ_n approximates τ. Clearly τ_n will have only integer slopes and $\tau_n \to \tau$ as $n \to \infty$ in the uniform norm.

The Frobenius-Perron operator P_{τ_n}, when restricted to step functions on J_n, can be represented by a matrix [6], which we denote by P_n; its entries are given by

$$\begin{aligned} p_{ij}^n &= 1/\tau_n'|_{I_i} \quad \text{if } I_j^n \subset \tau_n(I_i^n), \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

In [3] it is shown that P_n is similar to a stochastic matrix and therefore has a fixed point f_n, which we regard as a step function on J_n. Our aim is to prove that f_n converges weakly to f^*, the density of the unique measure invariant under τ.

DEFINITION 1. Let $\{\mu_n\}$ be a sequence of absolutely continuous probability measure on $(I, \mathfrak{B})$ and let f_n be the density of μ_n. We shall say that $f_n(\mu_n)$ converges weakly to the density f (measure μ) if and only if for each $g \in C$, the space of real, bounded and continuous functions on I,

$$\int_I g(x) f_n(x)\, m(dx) \to \int_I g(x) f(x) m(dx)$$

as $n \to \infty$.

In fact it is sufficient that g is in a space D dense in C [**4**, Theorem 12.2]. For our purposes, we shall use C^1, the space of functions on I which have continuous first derivatives.

DEFINITION 2. Let g be any function from I into $(-\infty, \infty)$, and let δ and ε be positive numbers. We denote by $\partial_{\delta,\varepsilon}(g)$ the set of those points $x \in I$ for which the distance between $g(x')$ and $g(x'')$ exceeds ε for some pair of points x', x'' in the open interval $(x-\delta, x+\delta)$.

A more general version of the following theorem is proved in [**5**].

THEOREM 2. *Let $\{g_n\}_{n\geq 1}$ be a sequence of bounded, real-valued and measurable functions defined on S and let α be a real number. Then a necessary and sufficient condition that $\int_I g_n(x) f_n(x)\, m(dx) \to \alpha$ for every sequence $\{f_n\}$ converging weakly to f is that*

(i) $\{g_n\}_{n\geq 1}$ *is uniformly bounded,*
(ii) $\int_I g_n(x) f(x)\, m(dx) \to \alpha$ *and*
(iii) $\forall \varepsilon > 0$, $\lim_{\delta\to 0} \limsup_{n\to\infty} \int_{\partial_{\delta,\varepsilon}(g_n)} f(x)\, m(dx) = 0$.

It can be shown that (iii) holds iff

(iii′) $\forall \varepsilon > 0$, for every sequence $\{\delta_k\}$ of positive numbers converging to 0, and for every subsequence $\{g_{n_k}\}$, $\int_{\cap_{k=1}^{\infty} \partial_{\delta_k,\varepsilon}(g_{n_k})} f(x)\, m(dx) = 0$.

LEMMA 1. *Let g be a bounded, piecewise continuous function on* $[0,1]$ *whose set of discontinuity points, D, has Lebesgue measure* 0. *Let $\{g_n\}$ be a uniformly bounded sequence of piecewise continuous functions which approaches g uniformly. Then, if $f_n \overset{\omega}{\to} f$,*

$$\int_I g_n(x) f_n(x)\, m(dx) \to \int_I g(x) f(x)\, m(dx)$$

as $n \to \infty$.

PROOF. Since $g_n \to g$ uniformly, we have

$$\int_I g_n(x) f(x)\, m(dx) \to \int_I g(x) f(x)\, m(dx) = \alpha.$$

It remains to prove (iii′). Let $\varepsilon > 0$. Then for any sequence $\{\delta_k\}$ of positive numbers converging to 0 and every subsequence $\{g_{n_k}\}$, $\cap_{k=1}^{\infty} \partial_{\delta_k,\varepsilon}(g_{n_k}) \subset D$. Since $m(D) = 0$, (iii′) is valid and Theorem 2 can be invoked. Q.E.D.

3. Main result.

LEMMA 2. *Let $\{\tau_n\}$ be a sequence of nonsingular transformations from $I \to I$ which approach τ uniformly. Let $f \in \mathcal{L}_1$. Then*

$$\int_I h(x)(P_{\tau_n} f)(x)\, m(dx) \to \int_I h(x)(P_\tau f)(x)\, m(dx)$$

as $n \to \infty$ for any $h \in C^1$.

PROOF. From the definition of the Frobenius-Perron operator, we have

$$\int_I h(x)\left(P_{\tau_n} f(x) - P_\tau f(x)\right) m(dx)$$
$$= \int_I h(x)\left\{\frac{d}{dx}\int_{\tau_n^{-1}[0,x]} f(y)\,m(dy) - \frac{d}{dx}\int_{\tau^{-1}[0,x]} f(y) m(dy)\right\} m(dx).$$

Integrating by parts,

$$\int_0^1 h(x)\left(\frac{d}{dx}\int_{\tau^{-1}[0,x]} f(y)\,m(dy)\right) m(dx)$$
$$= g(1)\int_0^1 f(y)\,m(dy) - \int_0^1 \int_{\tau^{-1}[0,x]} f(y)\,m(dy) g'(x)\,m(dx).$$

Thus,

$$\int_I h(x)\left(P_{\tau_n} f(x) - P_\tau f(x)\right) m(dx)$$
$$= \int_I \left\{\int_{\tau^{-1}[0,x]} f(y)\,m(dy) - \int_{\tau_n^{-1}[0,x]} f(y)\,m(dy)\right\} h'(x)\,m(dx)$$

and

$$\left|\int_I h(x)\left(P_{\tau_n} f(x) - P_\tau f(x)\right) m(dx)\right|$$
$$\leqslant \int_I \int_{(\tau^{-1}[0,x])\triangle(\tau_n^{-1}[0,x])} |f(y)|\,m(dy) h'(x)\,m(dx),$$

where $\triangle$ denotes the symmetric difference. Since $\tau_n \to \tau$ uniformly as $n \to \infty$, $m\{(\tau^{-1}[0, x])\triangle(\tau_n^{-1}[0, x])\} \to 0$ as $n \to \infty$. Since $h'(x)$ is continuous on I, it is bounded. This completes the proof. Q.E.D.

We can now state the main result of this note.

THEOREM 3. *Let τ: $I \to I$ be a nonsingular, piecewise continuous map, whose set of discontinuities has Lebesgue measure* 0, *and let τ admit a unique absolutely continuous probability measure μ. Let $\{\tau_n\}$ be a sequence of piecewise linear Markov maps which approach τ uniformly. Let f_n denote a fixed point of $P_n \equiv P_{\tau_n}$, where $\| f_n \| = 1$ and $f_n > 0$. Then $f_n \overset{\omega}{\to} f^*$ as $n \to \infty$, where f^* is the density function of μ.*

PROOF. Since I is compact, the family of probability measures $\{\mu_n\}$, defined by $\mu_n(E) = \int_E f_n(x)\,m(dx)$, is weakly compact. Hence there exists a subsequence $\{f_{n_i}\}$ and a function f: $I \to I$ such that $f_{n_i} \overset{\omega}{\to} f$.

Now, for any $h \in C^1$,

$$\left|\int_I h(x)(f(x) - P_\tau f(x))\,m(dx)\right|$$
$$\leqslant \left|\int_I h(x)(f(x) - f_{n_i}(x))\,m(dx)\right| + \left|\int_I h(x)(f_{n_i}(x) - P_{n_i} f_{n_i}(x))\,m(dx)\right|$$
$$+ \left|\int_I h(x)(P_{n_i} f_{n_i}(x) - P_{n_i} f(x))\,m(dx)\right| + \left|\int_I h(x)(P_{n_i} f(x) - P_\tau f(x))\,m(dx)\right|.$$

The first term approaches 0 since $f_{n_i} \xrightarrow{\omega} f$. Since $P_{n_i} f_{n_i} = f_{n_i}$, the second term is identically 0. The fourth term approaches 0 by virtue of Lemma 2. Consider now the third term,

$$\int_0^1 h(x) \frac{d}{dx}\left\{\int_{\tau_{n_i}^{-1}[0,x]} \left(f_{n_i}(y) - f(y)\right) m(dy)\right\} m(dx)$$

$$= \int_0^1 \left\{\int_{\tau_{n_i}^{-1}[0,x]} \left[f_{n_i}(y) - f(y)\right] m(dy)\right\} h'(x)\, m(dx).$$

Fix $x \in [0,1]$ and consider

$$A_{n_i}(x) \equiv \int_{\tau_{n_i}^{-1}[0,x]} \left[f_{n_i}(y) - f(y)\right] m(dy)$$

$$= \int_I \chi_{\tau_{n_i}^{-1}[0,x]}(y) f_{n_i}(y)\, m(dy) - \int_I \chi_{\tau_{n_i}^{-1}[0,x]}(y) f(y)\, m(dy).$$

Now $\chi_{\tau_{n_i}^{-1}[0,x]}(y)$ is a piecewise continuous step function which approaches $\chi_{\tau^{-1}[0,x]}$ uniformly as $n_i \to \infty$. Clearly

$$\int_I \chi_{\tau_{n_i}^{-1}[0,x]}(y) f(y)\, m(dy) \to \int_I \chi_{\tau^{-1}[0,x]}(y) f(y)\, m(dy) \equiv \alpha$$

as $n_i \to \infty$. Thus, it follows from Lemma 1 that $A_{n_i}(x) \to 0$ as $n_i \to \infty$. Note that $|A_{n_i}(x)| \leq 2$. Since $h \in C^1$, $|h'(x)| \leq L < \infty$. Hence, the Dominated Convergence Theorem implies that

$$\int_0^1 A_{n_i}(x) h'(x)\, m(dx) \to 0$$

as $n_i \to \infty$. We have, therefore, established, for any $h \in C^1$,

$$\int_I h(x)(f(x) - P_\tau(x))\, m(dx) = 0.$$

This means $P_\tau f(x) = f(x)$ m-a.e. But f^* is the unique fixed point of P_τ. Thus $f = f^*$ m-a.e., and $f_{n_i} \xrightarrow{\omega} f^*$. We have therefore shown that any weakly convergent subsequence of $\{f_n\}$ converges weakly to f. Hence $f_n \xrightarrow{\omega} f$ as $n \to \infty$. Q.E.D.

REMARKS. (1) Theorem 1 establishes a necessary condition for the existence of an absolutely continuous invariant for a general map $\tau: I \to I$.

(2) Classes of maps $\tau: I \to I$ which are nonexpanding and which have unique absolutely continuous invariant measures can be found in [7–**10**]. Theorem II.8.3 of [**10**] describes some of the results in [7].

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