

## DISTORTION AND COEFFICIENT ESTIMATION OF SCHLICHT FUNCTIONS

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**ABSTRACT.** Let

$$S(k) = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S, \lim_{\rho \rightarrow 1^-} \frac{(1-\rho)^2}{\rho} \max_{|z|=\rho} |f(z)| = \alpha_f \geq k > 0 \right\}.$$

In this paper, we prove that for  $f(z) \in S(k)$  the inequality

$$|a_n| \leq n \sqrt{\frac{P + \sqrt{(1+p)A^{1/2} - p}}{1+P}}$$

holds where  $p = k^2/(1-k^2)$  and  $1 \leq A < (1.0657)^8$ . This strengthens a recent result of Horowitz.

Let  $S = \{f(z) = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ is analytic and univalent in the unit disc } |z| < 1\}$  and let  $S(k) = \{f(z) \in S, \lim_{\rho \rightarrow 1^-} (1-\rho)^2/\rho \max_{|z|=\rho} |f(z)| = \alpha_f > k > 0\}$ .

(In fact for  $f(z) \in S$ , the  $\lim_{\rho \rightarrow 1^-}$  exists, but for this work it suffices to use the  $\limsup$ .)

The Bieberbach Conjecture asserts that if  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is in  $S$ , then  $|a_n| \leq n$  ( $n = 2, 3, \dots$ ). In other words the extremal function of the Bieberbach Conjecture is  $z/(1-z)^2$  for which  $\alpha_f = 1$ . The best approximation to the conjecture so far proved is due to D. Horowitz [1] who showed that

$$(1) \quad |a_n| < 1.0657n \quad (n \geq 2).$$

In this paper, we improve Horowitz's result (1). Besides, we prove a new theorem of distortion of schlicht functions. The method of [1] is used.

### 1. The main inequalities.

**THEOREM 1.** If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(k)$ , then

$$(1.1) \quad |a_n| \leq n \sqrt{\frac{P + \sqrt{(1+p)A^{1/2} - P}}{1+P}} < cn, \quad n = 2, 3, \dots,$$

where  $c = 1.0657$  [1],  $P = k^2/(1-k^2)$ ,  $A = \max\{\frac{209}{140}, c^8 - P[(\frac{209}{140})^{1/2} - \frac{7}{6}]^2\}$ .

This result is obviously stronger than (1).

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**THEOREM 2.** If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(k)$ , then

$$(1.2) \quad \left| \sum_{j=1}^L |a_j|^2 x_j \right|^2 + P \left| \sum_{j=1}^L (|a_j|^2 - j^2) x_j \right|^2 \leq \sum_{m,n=1}^L \sum_{K=1}^{n+m-1} \beta_k(n, m) |a_k|^2 x_m \bar{x}_n,$$

where

$$\beta_k(n, m) = \begin{cases} n - |k - m| & \text{for } |k - m| < n, \\ 0 & \text{otherwise,} \end{cases}$$

and  $P = k^2/(1 - k^2)$ .

**PROOF.** We know that [2]

$$(1.3) \quad \sum_{p,q=0}^m y_p \bar{y}_q \left\{ \left| \frac{f_p - f_q}{z_p - z_q} \right|^2 \frac{1}{|1 - z_p \bar{z}_q|^2} - \left| \frac{f_p f_q}{z_p z_q} \right|^2 \right\} \geq 0, \quad (f_p = f(z_p)).$$

In (1.3), set  $y_p = \lambda_p x_z$  ( $p \neq 0$ ),  $y_0 = x$ , and let

$$\begin{aligned} A_{1,2} &= \sum_{p=1}^m \lambda_p \left\{ \left| \frac{f_p - f_0}{z_p - z_0} \right|^2 \frac{1}{|1 - z_p \bar{z}_0|^2} - \left| \frac{f_p f_0}{z_p z_0} \right|^2 \right\}, \quad \bar{A}_{12} = A_{21}, \\ A_{2,2} &= \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \left\{ \left| \frac{f_p - f_q}{z_p - z_q} \right|^2 \frac{1}{|1 - z_p \bar{z}_q|^2} - \left| \frac{f_p f_q}{z_p z_q} \right|^2 \right\}, \\ A_{11} &= |f'_0|^2 \frac{1}{(1 - |z_0|^2)^2} - \left| \frac{f_0}{z_0} \right|^4. \end{aligned}$$

Then (1.3) may be written in the following form:

$$(1.4) \quad \sum_{i,j=1}^2 A_{i,j} x_i \bar{x}_j \geq 0.$$

An immediate consequence is

$$(1.5) \quad A_{1,1} A_{2,2} - |A_{1,2}|^2 \geq 0.$$

That is,

$$(1.6)$$

$$\begin{aligned} &\left\{ |f'_0|^2 \frac{1}{(1 - |z_0|^2)^2} - \left| \frac{f_0}{z_0} \right|^4 \right\} \left\{ \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \left( \left| \frac{f_p - f_q}{z_p - z_q} \right|^2 \frac{1}{|1 - z_p \bar{z}_q|^2} - \left| \frac{f_p f_q}{z_p z_q} \right|^2 \right) \right\} \\ &\geq \left| \sum_{p=1}^m \lambda_p \left\{ \left| \frac{f_p - f_0}{z_p - z_0} \right|^2 \frac{1}{|1 - z_p \bar{z}_0|^2} - \left| \frac{f_p f_0}{z_p z_0} \right|^2 \right\} \right|^2. \end{aligned}$$

Let  $\max_{|z|=\rho} |f(z)| = |f(\rho e^{i\theta_0})|$  and divide by  $|f_0/z_0|^4$  on both sides of (1.6). Next make  $z_0 \rightarrow e^{i\theta_0}$ , since

$$\left| \frac{z_0 f'_0}{f_0} \right| \leq \frac{1 + |z_0|}{1 - |z_0|} \quad \text{and} \quad k \leq \alpha_f = \lim_{\substack{z_0 \rightarrow e^{i\theta_0} \\ |z_0| < 1}} \frac{(1 - |z_0|)^2}{|z_0|} |f(z_0)|,$$

(1.6) can be reduced to

$$(1.7) \quad \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \left\{ \left| \frac{f_p - f_q}{z_p - z_q} \right|^2 \frac{1}{|1 - z_p \bar{z}_q|^2} - \left| \frac{f_p f_q}{z_p z_q} \right|^2 \right\} \left( \frac{1}{k^2} - 1 \right) \\ \geq \left| \sum_{p=1}^m \lambda_p \left( \frac{1}{|1 - z_p e^{i\theta_0}|^4} - \left| \frac{f_p}{z_p} \right|^2 \right) \right|^2.$$

The method of [2] gives

$$(1.8) \quad \left( \frac{1}{k^2} - 1 \right) \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \int_0^{2\pi} \int_0^{2\pi} \left\{ \left| \frac{f(\rho_p e^{i\theta}) - f(\rho_q e^{i\varphi})}{\rho_p e^{i\theta} - \rho_q e^{i\varphi}} \right|^2 \frac{1}{|1 - \rho_p \rho_q e^{i(\theta-\varphi)}|^2} \right. \\ \left. - \left| \frac{f(\rho_p e^{i\theta}) f(\rho_q e^{i\theta})}{\rho_p \rho_q} \right|^2 \right\} d\theta d\varphi \\ \geq \left| \sum_{p=1}^m \lambda_p \int_0^{2\pi} \left( \frac{1}{|1 - \rho_p e^{i(\theta-\theta_0)}|^4} - \left| \frac{f(\rho_p e^{i\theta})}{\rho_p} \right|^2 \right) d\theta \right|^2.$$

Inequality (1.8) can be converted to a statement about coefficients by the method of [2]. Thus inequality (1.8) implies inequality (1.2) and Theorem 2 is proved.

**THEOREM 3.** If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(k)$ , then

$$(1.9) \quad \left( \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \right)^2 + P \left( \sum_{K=1}^{2n} \Lambda_1(n, K) (|a_K|^2 - K^2) \right)^2 \\ \leq \sum_{K=1}^{4n} \Lambda_2(n, K) |a_K|^2,$$

$$(1.10) \quad \left( \sum_{K=1}^{4n} \Lambda_2(n, K) |a_K|^2 \right)^2 + P \left\{ \sum_{K=1}^{4n} \Lambda_2(n, K) (|a_K|^2 - K^2) \right\}^2 \\ \leq \sum_{K=1}^{8n} \Lambda_3(n, K) |a_K|^2,$$

where  $\Lambda_1(n, K) = n - |K - n|$ ,  $|n - K| < n$

$$\Lambda_m(n, K) = \sum_{j=\lfloor (K+1)/2 \rfloor}^{2^{m-1}n} \sum_{l=|K-j|}^j B(j, l) (l - |j - K|) \Lambda_{m-1}(n, j) \Lambda_{m-1}(n, l)$$

and  $B(j, l) = 1$ ,  $j = l$ ;  $2, j \neq l$  [1].

The proof of this theorem is similar to [1] except using inequality (1.2) in place of the Fitzgerald inequality.

**PROOF OF THEOREM 1.** In (1.2) set  $L = n$ ,  $x_1 = x_2 = \dots = x_{n-1} = 0$  and  $x_n = 1$  to obtain

$$(1.11) \quad |a_n|^4 + P(|a_n|^2 - n^2)^2 \leq \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2.$$

From (1.9) and (1.10), it follows that

$$(1.12) \quad \left( \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \right)^2 + P \left\{ \sum_{K=1}^{2n} \Lambda_1(n, K) (|a_K|^2 - K^2) \right\}^2 \leq \sum_{K=1}^{4n} \Lambda_2(n, K) |a_K|^2 \leq \sqrt{\sum_{K=1}^{8n} \Lambda_3(n, K) |a_K|^2} \leq c^8 n^8$$

where  $c \doteq 1.0657$  [1].

If  $\sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \leq \sqrt{209/140} n^4$ . Then

$$(1.13) \quad |a_n|^4 + P(|a_n|^2 - n^2)^2 \leq \sqrt{\frac{209}{140}} n^4$$

by (1.11). It follows that

$$(1.14) \quad |a_n| \leq n \sqrt{\frac{P + \sqrt{(1+P)\sqrt{209/140} - P}}{1+P}} < \left(\frac{209}{140}\right)^{1/8} n < cn.$$

On the other hand, if  $\sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \geq \sqrt{209/140} n^4$  then, since  $\sqrt{209/140} > \frac{7}{6}$  and  $\sum_{K=1}^{2n} \Lambda_1(n, K) K^2 < \frac{7}{6} n^4$  [2]

$$(1.15) \quad \left( \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \right)^2 + P \left( \sqrt{\frac{209}{140}} - \frac{7}{6} \right)^2 n^8 \leq \left( \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \right)^2 + P \left\{ \sum_{K=1}^{2n} \Lambda_1(n, K) (|a_K|^2 - K^2) \right\}^2 \leq c^8 n^8$$

by (1.12).

From (1.11) and (1.15),  $c^8 - p(\sqrt{209/140} - \frac{7}{6})^2 \geq 0$  and

$$(1.16) \quad |a_n|^4 + P(|a_n|^2 - n^2)^2 \leq \sum_{K=1}^{2n} \Lambda_1(n, K) |a_K|^2 \leq \sqrt{c^8 - P \left( \sqrt{\frac{209}{140}} - \frac{7}{6} \right)^2} n^4.$$

This gives

$$(1.17) \quad |a_n| \leq n \sqrt{\frac{P + \sqrt{(1+P)\{c^8 - P(\sqrt{209/140} - 7/6)^2\}^{1/2} - P}}{1+P}} < cn.$$

This completes the proof of Theorem 1.

## 2. Theorem of distortion.

**THEOREM 4.** If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ , then

$$(2.1) \quad |f(z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + |f(-z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ \leq \frac{|z|^2}{(1-|z|)^4} + \frac{|z|^2}{(1+|z|)^4},$$

$$(2.2) \quad |f'(z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + |f'(-z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ \leq \left\{ \frac{1+|z|}{(1-|z|)^3} \right\}^2 + \left\{ \frac{1-|z|}{(1+|z|)^3} \right\}^2,$$

where  $G_m(z) = 1/z^m + \bar{z}_m - F_m(1/f(z))$  and  $F_m(w)$  is  $m$ th order Faber's polynomial.

The proof of Theorem 4 is based on the following inequality [3]:

$$(2.3) \quad \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \left| \frac{f_p f_q}{z_p z_q} \right|^{\alpha} \exp \sum_{n=1}^{\infty} \frac{\alpha}{2n} G_n(z_p) \overline{G_n(z_q)} \leq \sum_{p,q=1}^m \lambda_p \bar{\lambda}_q \left| \frac{f_p - f_q}{z_p - z_q} \right|^{\alpha} \frac{1}{|1 - z_p \bar{z}_q|^{\alpha}}$$

where  $\alpha > 0$ .

In (2.3), set  $m = 2$ ,  $z_1 = z$ ,  $z_2 = -z$ ,  $\alpha = 1$  and  $\lambda_1 = \lambda_2 = 1$  to obtain

$$(2.4) \quad \left| \frac{f(z)}{z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + \left| \frac{f(-z)}{-z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ + 2 \left| \frac{f(z)f(-z)}{z^2} \right| R \exp \sum_{n=1}^{\infty} \frac{1}{2n} G_n(z) \overline{G_n(-z)} \\ \leq \frac{1}{1-|z|^2} \{ |f'(z)| + |f'(-z)| \} + \frac{1}{1+|z|^2} \left| \frac{f(z) - f(-z)}{z} \right|.$$

Next in (2.3), set  $z_1 = z$ ,  $z_2 = -z$ ,  $\alpha = 1$  and  $\lambda_1 = 1$ , also put  $\lambda_2 = -1$  to obtain

$$(2.5) \quad \left| \frac{f(z)}{z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + \left| \frac{f(-z)}{-z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ - 2 \left| \frac{f(z)f(-z)}{z^2} \right| R \exp \sum_{n=1}^{\infty} \frac{1}{2n} G_n(z) \overline{G_n(-z)} \\ \leq \frac{1}{1-|z|^2} \{ |f'(z)| + |f'(-z)| \} - \frac{1}{1+|z|^2} \left| \frac{f(z) - f(-z)}{z} \right|.$$

Adding (2.3), (2.4) we have

$$(2.6) \quad \left| \frac{f(z)}{z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + \left| \frac{f(-z)}{-z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ \leq \frac{1}{1 - |z|^2} (|f'(z)| + |f'(-z)|).$$

Goluzin proved that [4]

$$(2.7) \quad |f'(z)| + |f'(-z)| \leq \frac{1 + |z|}{(1 - |z|)^3} + \frac{1 - |z|}{(1 + |z|)^3}.$$

The inequality (2.1) follows from (2.6) and (2.7). To prove (2.2), we observe that

$$(2.8) \quad (1 - |\xi|^2)^2 \left| \frac{f'(\xi)}{f(\xi)} \xi \right|^2 + \left| \frac{\xi}{f(\xi)} \right|^2 \leq (1 + |\xi|)^4 + (1 - |\xi|)^4.$$

This inequality is easily deduced by Goluzin's method [4] and it is omitted here. Inequality (2.8) may be written in the form

$$(2.9) \quad (1 - |\xi|^2)^2 |f'(\xi)|^2 + 1 \leq \{(1 - |\xi|)^4 + (1 + |\xi|)^4\} \left| \frac{f(\xi)}{\xi} \right|^2.$$

In (2.9), set  $\xi = z, -z$  and multiply both sides of (2.6) by  $\exp \sum_{n=1}^{\infty} (1/2n) |G_n(z)|^2$ ,  $\exp \sum_{n=1}^{\infty} (1/2n) |G_n(-z)|^2$  respectively. Adding these results we obtain

$$(2.10) \quad (1 - |z|^2)^2 \left\{ |f'(z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + |f'(-z)|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \right\} \\ + \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 + \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \\ \leq \{(1 - |z|)^4 + (1 + |z|)^4\} \left\{ \left| \frac{f(z)}{z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(z)|^2 \right. \\ \left. + \left| \frac{f(-z)}{-z} \right|^2 \exp \sum_{n=1}^{\infty} \frac{1}{2n} |G_n(-z)|^2 \right\} \\ \leq \{(1 - |z|)^4 + (1 + |z|)^4\} \left\{ \frac{1}{(1 - |z|)^4} + \frac{1}{(1 + |z|)^4} \right\}$$

by (2.1).

Since  $\exp \sum_{n=1}^{\infty} (1/2n) |G_n(z)|^2 + \exp \sum_{n=1}^{\infty} (1/2n) |G_n(-z)|^2 \geq 2$  (2.2) follows at once from (2.10). Thus the proof of this distortion theorem is complete.

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