PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

PERIODIC POINTS AND TOPOLOGICAL ENTROPY OF MAPS OF THE CIRCLE

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ABSTRACT. Let f be a continuous map from the circle to itself, let $P(f)$ denote the set of integers n for which f has a periodic point of period n. In this paper it is shown that the two smallest numbers in $P(f)$ are either coprime or one is twice the other.

1. Introduction. Let f be a continuous map of the circle into itself, let $P(f)$ denote the set of positive integers n such that f has a periodic point of (least) period n. If $P(f)$ does not consist of a single point, let p_1 and p_2 denote, respectively, the smallest and second smallest elements of $P(f)$. It will be shown that either p_1 and p_2 are coprime or $p_2 = 2p_1$.

This result can then be combined with results in [**1**, **3** and **6**] to prove

THEOREM 1. *Let $f \in C^0(S^1, S^1)$. Suppose that $P(f)$ contains more than one element. Let p_1 and p_2 denote the smallest elements of $P(f)$, with $p_1 < p_2$.*

If $2p_1 \neq p_2$ then:

(1) *p_1 and p_2 are coprime*;

(2) *$\alpha p_1 + \beta p_2 \in P(f)$ where α and β are any positive integers*;

(3) *The topological entropy of f, $h(f) \geq \log \mu_{p_1,p_2}$ where μ_{p_1,p_2} is the largest zero of $x^{p_1+p_2} - x^{p_2} - x^{p_1} - 1$.*

(4) *There exists a map $f_{p_1,p_2} \in C^0(S^1, S^1)$ such that*

$$P(f_{p_1,p_2}) = \{\alpha p_1 + \beta p_2 \mid \alpha \in N^+, \beta \in N^+\} \cup \{p_1, p_2\}$$

and $h(f_{p_1,p_2}) = \log \mu_{p_1,p_2}$.

If $2p_1 = p_2$ there exists a map, f_{p_1,p_2}, with $P(f_{p_1,p_2}) = \{p_1, p_2\}$ and $h(f_{p_1,p_2}) = 0$.

2. In this section the following theorem is proved.

THEOREM 2.1. *Let $f \in C^0(S^1, S^1)$. Suppose that $P(f)$ is not a singleton. Let p_1, p_2 denote the two smallest elements of $P(f)$. Then either p_1 and p_2 are coprime or $p_2 = 2p_1$.*

The theorem is trivially true if $p_1 = 1$, so throughout this section it will be assumed that f has no fixed points.

DEFINITION 2.2. Let f be an endomorphism of the circle of degree 1 and let F be a lifting of f. The *rotation number* $\rho(F, x)$ is defined by $\rho(F, x) = \limsup_{n\to\infty}(1/n)(F^n(x) - x)$, and the rotation set $\rho(F) = \{\rho(F, x) : x \in \mathbf{R}\}$.

Received by the editors March 8, 1982 and, in revised form, June 14, 1982.
1980 *Mathematics Subject Classification*. Primary 58F20.

The rotation set $\rho(F)$ is a closed interval or a single point, and a different lifting of f just translates the rotation set by an integer (see [**7** and **4** or **8**]).

In [**4** and **8**] the following is shown.

LEMMA 2.3. *Let $f \in C^0(S^1, S^1)$ be a degree one map with rotation interval $[a, b]$. Then for any rational number $m/n \in [a, b]$, with m and n coprime, n belongs to $P(f)$.*

LEMMA 2.4. *Let a/b, c/d be two rational numbers contained in the interval $[0, 1]$. Suppose that $a/b < c/d$ and that b and d have a common factor. Then there exists a rational number m/n satisfying $a/b \leqslant m/n \leqslant c/d$, such that:*

(i) $n < \max(b, d)$;

(ii) $n \notin \{b, d\}$.

PROOF. The proof will be divided into two cases depending on whether the fractions a/b, c/d are expressed in lowest terms or not.

Case 1. Suppose that both a/b and c/d are already in lowest terms, i.e. the numerator and denominator are coprime. Then both a/b and c/d will occur in the $\max(b, d)$ row of the Farey series. By elementary number theory there exists a rational number m/n, with required properties (see, for example, [**5**]).

Case 2. Suppose that a/b and c/d are not already in lowest terms. Cancellation either gives the required result immediately or reduces to the first case.

PROOF OF THEOREM 2.1. Since f has no fixed points it must have degree one. Thus the rotation set is defined and, without loss of generality, may be assumed to be contained in the unit interval $[0, 1]$.

Choose $x \in S^1$ such that $f^{p_1}(x) = x$ and choose $y \in S^1$ such that $f^{p_2}(y) = y$.

Suppose that p_1 and p_2 have a common factor. Then write $p_1 = kq$ and $p_2 = lq$ where k and l are coprime.

Let $\rho(x) = a/kq$ and $\rho(y) = b/lq$. Clearly $(a, kq) = 1$, otherwise Lemma 2.3 would imply the existence of a periodic point with period smaller than p_1.

Suppose that $a/kq \neq b/lq$. Then applying Lemma 2.4 and then Lemma 2.3 shows that there exists a point of period n, where $n \neq p_1$ and $n < p_2$. This contradicts the definition of p_1 and p_2.

Thus $a/kq = b/lq$ and so $bk = al$. Since $(k, l) = 1$, k divides a; but $(a, kq) = 1$ and so $k = 1$.

It has been shown that if p_1 and p_2 are not coprime then $p_2 = lp_1$ and $\rho(x) = \rho(y)$.

Now consider the map f^{p_1}. This has a fixed point x, and y is a point of period l. Clearly 1 and l are the two smallest elements of $P(f^p)$. Since f^p is of degree one there exists a lifting g such that $\rho(x) = \rho(y) = 0$.

Thus $g \in C^0(\mathbf{R}, \mathbf{R})$ and 1 and l are the two smallest elements of $P(g)$, (if a lifting of a degree one map has a periodic point of period k, then so does the map). Sarkovskii's theorem then shows that $l = 2$.

3. Proof of Theorem 1. Louis Block has extensively studied the case when $p_1 = 1$. When $p_1 = 1$, Theorem 1 is weaker then the results in [**2** and **3**].

Ito [**6**] has shown the following:

THEOREM 3.1. *Let $f \in C^0(S^1, S^1)$. Let $m, n \in P(f)$ such that $m \geq 2$, $n \geq 2$ and $(m, n) = 1$. Then $h(f) \geq \log \mu_{m,n}$ where $\mu_{m,n}$ is the largest zero of $x^{m+n} - x^m - x^n - 1$.*

In [1] the following is proved.

THEOREM 3.2. *Let $f \in C^0(S^1, S^1)$. Let p_1, p_2 be the two smallest elements of $P(f)$. If $(p_1, p_2) = 1$, then $\alpha p_1 + \beta p_2 \in P(f)$ for any positive integers α and β.*

Thus statements (1), (2), and (3) of Theorem 1 are true.

To complete the proof it is only necessary to construct maps f_{p_1,p_2}, with minimal entropy and with minimal number of periodic points.

When $(p_1, p_2) = 1$, Ito [6] constructs a map $f_{p_1,p_2}\colon S^1 \to S^1$. By looking at the associated A-graph he shows that $h(f_{p_1,p_2}) = \log \mu_{p_1,p_2}$. The A-graph also shows that $P(f_{p_1,p_2}) = \{\alpha p_1 + \beta p_2 \mid \alpha \in \mathbf{N}^+, \beta \in \mathbf{N}^+\} \cup \{p_1, p_2\}$.

Now consider the case $p_2 = 2p_1$. Let $S^1 = \mathbf{R}/\mathbf{Z}$ and let $f_{1,2}\colon S^1 \to S^1$ be the map induced from $F_{1,2}\colon \mathbf{R} \to \mathbf{R}$ defined by

$$F_{1,2}(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{3}, \\ -x + 1, & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ 2x - 1, & \frac{2}{3} \leq x \leq 1, \end{cases}$$

and $F(x + k) = F(x)$ for $k \in \mathbf{Z}$.

It is easily checked that $f_{1,2}$ has only periodic points of periods 1 and 2.

Similarly, let $f_{p,2p}\colon S^1 \to S^1$ be the map induced from $F_{p,2p}\colon \mathbf{R} \to \mathbf{R}$ defined by

$$F_{p,2p}(x) = \begin{cases} 2x + 1/p, & 0 \leq x \leq 1/3p, \\ -x + 2/p, & 1/3p \leq x \leq 2/3p, \\ 2x, & 2/3p \leq x \leq 1/p, \\ x + 1/p, & 1/p \leq x \leq 1, \end{cases}$$

and $F_{p,2p}(x + k) = F_{p,2p}(x)$ for $k \in \mathbf{Z}$. This map has only periodic points of period p and $2p$. Clearly $h(f_{p,2p}) = 0$.

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