PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

TOTALLY ANALYTIC SPACES UNDER $V = L$

ZOLTÁN BALOGH AND HEIKKI JUNNILA

ABSTRACT. The following results obtain under the axiom of constructibility ($V = L$):

Assume that every subset of a topological space X is analytic. Then X is σ-left-separated. Moreover, if the character of X is $\leq \omega_1$, then X is σ-discrete.

Assume that X is a perfectly normal space of character $\leq \omega_1$ such that every subset of X belongs to the σ-algebra generated by the analytic subsets of X. Then X is σ-discrete.

1. Introduction. By an *analytic* subset of a topological space X, we mean a set that can be obtained from a family of closed subsets of X by the A-operation, i.e., a set of the form $\bigcup_{\phi \in {}^\omega\omega} \bigcap_{n \in \omega} F_{\phi|n}$, where the sets F_s, $s \in {}^{<\omega}\omega$, are closed in X. We say that X is a *totally analytic* space if each subset of X is analytic. This notion generalizes that of a space in which each subset is an F_σ-set.

A space X is *σ-discrete* (*F_σ-discrete*) if X is the countable union of discrete (and closed) subspaces. X is *left-separated* if there exists a well-order $<$ on X such that all the initial segments $\{y \in X: y < x\}$, $x \in X$, are closed in X. We say that X is *σ-left-separated* if X is the countable union of left-separated subspaces. Note that the concept of an F_σ-left-separated space is superfluous, since such a space is actually left-separated; in particular, every F_σ-discrete space is left-separated.

The *character* of a space X is the least cardinal number κ such that every point of X has a neighbourhood-base consisting of at most κ sets; the character of X is denoted by $\chi(X)$.

The results in this paper give some partial answers to the question whether it is consistent with ZFC that every totally analytic space is σ-discrete. We can only aim for a consistency result since it follows from MA + $\neg$CH that $\mathbf{R}$ has uncountable (hence non-σ-discrete) subspaces that are totally analytic (and in fact, with each subset an F_σ-set). To obtain our results, we use Gödel's constructibility axiom $V = L$. Previous to the results below, G. M. Reed has shown that, under $V = L$, each first countable normal space with every subset an F_σ-set is F_σ-discrete [**8**], and R. W. Hansell has shown that, under $V = L$, a totally analytic space X of character $\leq \omega_1$ is σ-discrete provided that the product of X with the irrationals is normal [**5**]. Like Reed and Hansell, we rely on W. G. Fleissner's techniques and results in [**2**] (dealing with the seemingly unrelated topic of collectionwise Hausdorffness of normal spaces of character $\leq \omega_1$ under $V = L$).

Received by the editors November 23, 1981 and, in revised form, May 25, 1982.
1980 *Mathematics Subject Classification*. Primary 54H05, 04A15, 28A05; Secondary 54A25, 54A35.
Key words and phrases. $V = L$, analytic set, σ-discrete, left-separated, character.

0002-9939/82/0000-0600/$02.50

2. The main theorems. We have only been able to obtain σ-discreteness from $V = L$ for totally analytic spaces under the restriction that the spaces under consideration have character $\leqslant \omega_1$; for general spaces, we have the following result:

THEOREM 2.1 ($V = L$). *Each totally analytic space is σ-left-separated.*

The proof of Theorem 2.1 is a modification of Fleissner's proof of the main result in [2]. We shall give some lemmas from which the theorem follows; for a few of these lemmas, we do not give a proof but only indicate the changes that are needed in the proofs of the corresponding results in [2].

In [2], mappings $A \to \omega_1$ were used to code open separations of subsets of a discrete subset A of a space X (of character $\leqslant \omega_1$). Here we use mappings $X \to {}^{\omega}\omega$ to code partitions of X into analytic sets. (Note that, under $V = L$, $|{}^{\omega}\omega| = \omega_1$.)

LEMMA 2.2. *Let $\mathcal{C}$ be a disjoint family of analytic subsets of X. Then there exists $f\colon X \to {}^{\omega}\omega$ such that, for each $C \in \mathcal{C}$,*

$$C = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} \overline{\{x \in C\colon f(x) \mid n = \phi \mid n\}}\,.$$

PROOF. For every $C \in \mathcal{C}$, let $\{F_s(C)\colon s \in {}^{<\omega}\omega\}$ be a family of closed subsets of X such that $C = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} F_{\phi|n}(C)$. Define $f\colon X \to {}^{\omega}\omega$ as follows. If $x \in X \sim \bigcup \mathcal{C}$, let $f(x)$ be an arbitrary element of ${}^{\omega}\omega$. If $x \in C \in \mathcal{C}$, let $f(x) \in {}^{\omega}\omega$ be such that $x \in \bigcap_{n \in \omega} F_{f(x)|n}(C)$. That f has the required property follows by observing that for all $C \in \mathcal{C}$, $\phi \in {}^{\omega}\omega$ and $n \in \omega$, $\{x \in C\colon f(x) \mid n = \phi \mid n\} \subset F_{\phi|n}(C)$. $\square$

A connection between mappings $X \to {}^{\omega}\omega$ and σ-left-separatedness of X is given in the following lemma. Call X *A-left-separated* if there exists a well-order $<$ on X and a map $f\colon X \to {}^{\omega}\omega$ such that, for each $x \in X$,

$$x \notin \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} \overline{\{y < x\colon f(y) \mid n = \phi \mid n\}}\,.$$

LEMMA 2.3. (i) *Every A-left-separated space is σ-left-separated.*

(ii) *If a space has a countable cover by analytic left-separated subspaces then the space is A-left-separated.*

PROOF. (i) Assume that X, $<$ and f satisfy the condition defining A-left-separatedness. For each $x \in X$, let $n(x) \in \omega$ be such that

$$x \notin \overline{\{y < x\colon f(y) \mid n(x) = f(x) \mid n(x)\}}\,.$$

For each $s \in {}^{<\omega}\omega$, let $L_s = \{x \in X\colon f(x) \mid n(x) = s\}$. Then the sets L_s, $s \in {}^{<\omega}\omega$, cover X, and it is easy to check that these sets are left-separated as subspaces of X.

(ii) Assume that X has a cover $\{A_k\colon k < \omega\}$ by analytic subsets and that, for every $k < \omega$, there exists a well-ordering $<_k$ of A_k such that, for each $x \in A_k$, $x \notin \overline{\{y \in A_k\colon y <_k x\}}$ For each $k \in \omega$, let $\{F_s^k\colon s \in {}^{<\omega}\omega\}$ be a family of closed subsets of X such that $A_k = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} F^k_{\phi|n}$ and, for all $\phi \in {}^{\omega}\omega$ and $n \in \omega$, $F^k_{\phi|n+1} \subset F^k_{\phi|n}$. For each $x \in X$, let $k(x)$ be the least $k \in \omega$ such that $x \in A_k$, and let $f(x) \in {}^{\omega}\omega$ be such that $x \in \bigcap_{n \in \omega} F^{k(x)}_{f(x)|n}$. Define a well-order $\prec$ on X by setting $y \prec x$ iff either $k(y) < k(x)$ or $k(y) = k(x)$ and $y <_k x$. To show that X is

A-left-separated with respect to f and $\prec$, let $x \in X$ and let $k = k(x)$. For all $\phi \in {}^{\omega}\omega$ and $n \in \omega$, we have $\{y \prec x: f(y) \mid n = \phi \mid n\} = \bigcup_{j<k}\{y \in A_j: f(y) \mid n = \phi \mid n\} \cup \{y \in A_k: y <_k x \text{ and } f(y) \mid n = \phi \mid n\}$, and it follows that

$$\overline{\{y \prec x: f(y) \mid n = \phi \mid n\}} \subset \bigcup_{j<k} F^j_{\phi|n} \cup \overline{\{y \in A_k: y <_k x\}} \cap F^k_{\phi|n}.$$

Further, since the sets $F^i_{\phi|n}$, $n \in \omega$, are decreasing, it follows that

$$\begin{aligned}\bigcap_{n\in\omega} \overline{\{y \prec x: f(y) \mid n = \phi \mid n\}} &\subset \bigcup_{j<k}\left(\bigcap_{n\in\omega} F^j_{\phi|n}\right) \cup \overline{\{y \in A_k: y <_k x\}} \cap \bigcap_{n\in\omega} F^k_{\phi|n} \\ &\subset \bigcup_{j<k} A_j \cup \overline{\{y \in A_k: y <_k x\}} \cap A_k \\ &= \bigcup_{j<k} A_j \cup \{y \in A_k: y <_k x\} = \{y \in X: y \prec x\}.\end{aligned}$$

Consequently, $\bigcup_{\phi\in{}^{\omega}\omega} \bigcap_{n\in\omega} \overline{\{y \prec x: f(y) \mid n = \phi \mid n\}} \subset \{y: y \prec x\}$; since $x \notin \{y: y \prec x\}$, we have completed the proof of (ii). □

REMARK. By Lemma 2.3, a totally analytic space is σ-left-separated iff the space is A-left-separated. Thus we can prove Theorem 2.1 by showing that, under $V = L$, each totally analytic space is A-left-separated. Note that, corresponding to the similarly marked result on p. 295 of [**2**], we have the following consequence of Lemma 2.3:

(*) If a totally analytic space is the countable union of A-left-separated subspaces, then the space is A-left-separated.

For each cardinal number κ, denote by TALS(κ) (TALS($< \kappa$)) the statement that each totally analytic space of cardinality κ (of cardinality $< \kappa$) is A-left-separated. For $\kappa \leqslant \omega$, TALS(κ) holds trivially. We shall prove Theorem 2.1 by using transfinite induction on κ to show that TALS(κ) holds for every uncountable cardinal κ. For a singular cardinal κ of cofinality ω, it follows from (*) that TALS($< \kappa$) $\Rightarrow$ TALS(κ). Hence it suffices to prove that TALS($< \kappa$) $\Rightarrow$ TALS(κ) whenever κ is either regular or singular of cofinality $> \omega$.

Without loss of generality, we may assume that the underlying set of our totally analytic space X is a cardinal number κ; we shall make this assumption for the rest of the proof of Theorem 2.1.

For all $f: \kappa \to {}^{\omega}\omega$ and $\alpha < \kappa$, let $W^f_\alpha = \bigcup_{\phi\in{}^{\omega}\omega} \bigcap_{n\in\omega} \overline{\{\beta < \alpha: f(\beta) \mid n = \phi \mid n\}}$. The following result corresponds to Lemma 1 of [**2**]:

LEMMA 2.4. *Assume* TALS($< \kappa$). *Either* $A_f = \{\alpha < \kappa: W^f_\alpha \neq \alpha\}$ *is stationary for each* $f: \kappa \to {}^{\omega}\omega$ *or* X *is* A*-left-separated.*

PROOF. Assume that there exists $f: \kappa \to {}^{\omega}\omega$ and a cub $C \subset \kappa$ such that $C \cap A_f = \varnothing$. For each $\alpha \in \kappa$, let $\gamma(\alpha)$ be the least element of the set $\{\gamma \in C: \alpha < \gamma\}$. For every $\gamma \in C$, it follows from TALS($< \kappa$) that there exists a well-order $<_\gamma$ on γ and a map $f_\gamma: \gamma \to {}^{\omega}\omega$ such that for each $\alpha \in \gamma$, $\bigcup_{\phi\in{}^{\omega}\omega} \bigcap_{n\in\omega} \overline{\{\beta <_\gamma \alpha: f_\gamma(\beta) \mid n = \phi \mid n\}} \cap \gamma = \{\beta \in \gamma: \beta <_\gamma \alpha\}$. Define a well-order $\prec$ on κ by setting $\alpha \prec \beta$ iff either $\gamma(\alpha) < \gamma(\beta)$ or $\gamma(\alpha) = \gamma(\beta)$ and $\alpha <_{\gamma(\alpha)} \beta$. Define a binary operation $\updownarrow$ on ${}^{\omega}\omega$ by the condition that $\theta \updownarrow \psi(2n) = \theta(n)$ and $\theta \updownarrow \psi(2n + 1) = \psi(n)$ for all $\theta, \psi \in {}^{\omega}\omega$ and $n \in \omega$. Define

a map $g\colon \kappa \to {}^{\omega}\omega$ by setting $g(\alpha) = f \updownarrow f_{\gamma(\alpha)}$ for each $\alpha \in \kappa$. We show that X is A-left-separated with respect to $\prec$ and g. Let $\alpha \in \kappa$, and let $\gamma = \gamma(\alpha)$. Denote by δ the largest element of the set $\{\rho \in C \cup \{0\}\colon \rho \leqslant \alpha\}$, and let $B = \{\beta \in \gamma \sim \delta\colon \beta <_\gamma \alpha\}$; note that $\{\beta \in \kappa\colon \beta \prec \alpha\} = \delta \cup B$ and that, for each $\beta \in B$, $\gamma(\beta) = \gamma$. Let $\phi \in {}^{\omega}\omega$. To show that $\bigcap_{n\in\omega} \overline{\{\beta \prec \alpha\colon g(\beta) \mid n = \phi \mid n\}} \subset \{\beta\colon \beta \prec \alpha\}$, let $\theta \in {}^{\omega}\omega$ and $\psi \in {}^{\omega}\omega$ be such that $\phi = \theta \updownarrow \psi$. Then, for each $n \in \omega$, $\{\beta \prec \alpha\colon g(\beta) \mid 2n = \phi \mid 2n\} = \{\beta \prec \alpha\colon f(\beta) \mid n = \theta \mid n$ and $f_{\gamma(\beta)}(\beta) \mid n = \psi \mid n\} \subset \{\beta \in \delta\colon f(\beta) \mid n = \theta \mid n\} \cup \{\beta \in B\colon f(\beta) \mid n = \theta \mid n$ and $f_\gamma(\beta) \mid n = \psi \mid n\}$. It follows that

$$\bigcap_{n\in\omega} \overline{\{\beta \prec \alpha\colon g(\beta) \mid n = \phi \mid n\}} \subset \bigcap_{n\in\omega} \overline{\{\beta \in \delta\colon f(\beta) \mid n = \theta \mid n\}}$$
$$\cup \left[\bigcap_{n\in\omega} \overline{\{\beta \in B\colon f_\gamma(\beta) \mid n = \psi \mid n\}} \cap \bigcap_{n\in\omega} \overline{\{\beta \in B\colon f(\beta) \mid n = \theta \mid n\}} \right];$$

consequently, as $\delta, \gamma \in \kappa \sim A_f$ and $B \subset \{\beta \in \gamma\colon \beta <_\gamma \alpha\}$,

$$\bigcap_{n\in\omega} \overline{\{\beta \prec \alpha\colon g(\beta) \mid n = \phi \mid n\}} \subset \delta \cup \left[\bigcap_{n\in\omega} \overline{\{\beta <_\gamma \alpha\colon f_\gamma(\beta) \mid n = \psi \mid n\}} \cap \gamma \right]$$
$$\subset \delta \cup \{\beta \in \gamma\colon \beta <_\gamma \alpha\} = \{\beta \in \kappa\colon \beta \prec \alpha\}. \qquad \square$$

LEMMA 2.5 ($V = L$). *For a regular* κ, TALS($< \kappa$) $\Rightarrow$ TALS(κ).

PROOF. If we use Lemma 2.4 above to substitute for Lemma 1 of [2], the proof of Lemma 2.5 is almost identical to that of Lemma 3 of [2]. We assume that TALS($< \kappa$) holds but there is a totally analytic space X with underlying set κ such that X is not A-left-separated. Then the sets A_f of Lemma 2.4 are all stationary. It is easily seen that for all $f, g\colon \kappa \to {}^{\omega}\omega$ and $\alpha \in \kappa$, we have $A_f \cap (\alpha + 1) = A_g \cap (\alpha + 1)$ whenever $f \mid \alpha = g \mid \alpha$. Consequently, the sets A_f, for $f\colon \kappa \to {}^{\omega}\omega$, form what Fleissner calls a stationary system, and we can apply Lemma 2 of [2] to conclude that there exists $S \subset \kappa$ and $\Phi(\alpha)\colon \alpha \to {}^{\omega}\omega$, for $\alpha \in S$, such that for each $f\colon \kappa \to {}^{\omega}\omega$, the set $\{\alpha\colon \Phi(\alpha) = f \mid \alpha\}$ is a stationary set contained in A_f.

Whenever, $E \subset \kappa$, f is a map $E \to {}^{\omega}\omega$ and $B \subset E$, write

$$f^{\#}(B) = \bigcup_{\phi\in{}^{\omega}\omega} \bigcap_{n\in\omega} \overline{\{\beta \in B\colon f(\beta) \mid n = \phi \mid n\}}.$$

The proof of Lemma 3 of [2] works here once we make the following changes: for both U_α^f and $\overline{U_\alpha^f}$ use the set $f^{\#}(\alpha \cap H)$; for V_α^f and $\overline{V_\alpha^f}$ use $f^{\#}(\alpha \cap K)$; for W_α^f and $\overline{W_\alpha^f}$ use $f^{\#}(\alpha)$; for the notion "$f^{\#}$ separates H and K" use the condition $f^{\#}(H) \cap f^{\#}(K) = \varnothing$. $\square$

The changes indicated in the above proof also work to show that the results corresponding to Lemmas 4 and 5 of [2] hold in the present situation; in particular, we have the following result:

LEMMA 2.6 (GCH). *For a singular* κ *of cofinality* $> \omega$, TALS($< \kappa$) $\Rightarrow$ TALS(κ).

The above results complete the proof of Theorem 2.1. Note that since each F_σ-left-separated space is left-separated, we have the following consequence of Theorem 2.1.

COROLLARY 2.7 ($V = L$). *If every subset of a topological space is an F_σ-set, then the space is left-separated.*

Every uncountable left-separated space contains a nonseparable subspace; hence Theorem 2.1 gives the following result.

COROLLARY 2.8 ($V = L$). *Every totally analytic hereditarily separable space is countable.*

In light of Theorem 2 of [**1**], and Theorem 1 of [**1**], Theorem 4 of [**4**] or Corollary 4.9 of [**6**], the following is a consequence of Theorem 2.1.

COROLLARY 2.9 ($V = L$). *If a totally analytic space is either symmetrizable or semistratifiable, then the space is F_σ-discrete.*

Our second theorem indicates one more particular case, beyond those given in the above corollary, in which the conclusion of Theorem 2.1 can be strengthened to σ-discreteness. Whereas the proof of Theorem 2.1 was a modification of Fleissner's proof of the theorem in [**2**], the proof below consists of an application of that theorem itself. We also use a lemma from [**5**] on the representation of analytic sets.

THEOREM 2.10 ($V = L$). *Each totally analytic space of character $\leqslant \omega_1$ is σ-discrete.*

PROOF. Let X be a totally analytic space with $\chi(X) \leqslant \omega_1$. To show that X is σ-discrete, we construct a normal space Y with $\chi(Y) \leqslant \omega_1$; $V = L$ and Fleissner's theorem imply that Y is collectionwise Hausdorff, and from this we infer that X is σ-discrete.

Let $\mathcal{H}$ be the filter consisting of all sets $H \subset {}^{<\omega}\omega$ such that there exists $\phi \in {}^{\omega}\omega$ such that for each $\psi \in {}^{\omega}\omega$, if $\psi \geqslant \phi$, then there is $n \in \omega$ such that $\{\psi \mid m\colon m \geqslant n\} \subset H$ (these sets are called "final segments of ${}^{<\omega}\omega$" in [**5**]). Let $Y = X \times X \times {}^{<\omega}\omega$. For each $L \subset X$, denote by L^* the subset $\{\langle x, x, \varnothing\rangle\colon x \in L\}$ of Y. For all $H \in \mathcal{H}$ and $x \in X$, and for each map g from H into the set of all neighborhoods of x in X, let

$$U(x, g) = \{\langle x, x, \varnothing\rangle\} \cup (\{x\} \times X \times H) \cup \bigcup_{s \in H} (g(s) \times \{x\} \times \{s\}).$$

Topologize Y by making the points of $Y \sim X^*$ isolated and by using the sets $U(x, g)$ to define a neighborhood base for $\langle x, x, \varnothing\rangle \in X^*$.

Since $|{}^{<\omega}\omega| = \omega$, $|\mathcal{H}| \leqslant 2^\omega$, $\chi(X) \leqslant \omega_1$, and $2^\omega = \omega_1$ (under $V = L$), it is easily seen that $\chi(Y) \leqslant \omega_1$. To see that Y is normal, note first that X^* is a closed discrete subset of Y with discrete complement. Consequently, Y is normal provided that for each $L \subset X$, the sets L^* and $(X \sim L)^*$ can be separated by open subsets of Y. Let $L \subset X$. Since X is totally analytic, there are closed subsets F_s and K_s, $s \in {}^{<\omega}\omega$, of X such that $L = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} F_{\phi|n}$ and $X \sim L = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} K_{\phi|n}$; by Lemma 2 of [**5**], we can choose the sets F_s and K_s so that for all $\phi, \psi \in {}^{\omega}\omega$ and $n \in \omega$, $F_{\phi|n+1} \subset F_{\phi|n}$ and $K_{\phi|n+1} \subset K_{\phi|n}$ and, if $\phi \leqslant \psi$, then $F_{\phi|n} \subset F_{\psi|n}$ and $K_{\phi|n} \subset K_{\psi|n}$. For each $x \in X$, let $H_x = \{s \in {}^{<\omega}\omega\colon x \in F_s \sim K_s\}$ and $J_x = \{s \in {}^{<\omega}\omega\colon x \in K_s \sim F_s\}$; note that if $x \in L$, then $H_x \in \mathcal{H}$ and if $x \in X \sim L$, then $J_x \in \mathcal{H}$. For each $s \in {}^{<\omega}\omega$, let $g(s) = X \sim K_s$ and $f(s) = X \sim F_s$. Let $V = \bigcup_{x \in L} U(x, g \mid H_x)$ and $W = \bigcup_{x \in X \sim L} U(x, f \mid J_x)$, and note that these are open subsets of Y containing L^* and

$(X \sim L)^*$, respectively. We show that $V \cap W = \varnothing$. Assume on the contrary that there are $x \in L$, $z \in X \sim L$ and $\langle u, v, s\rangle \in U(x, g \mid H_x) \cap U(z, f \mid J_x)$. We have $s \in H_x \cap J_x$ and it follows that $x \in F_s$ and $z \in K_s$. Note that either $u = x$ or $u = z$. If $u = x$, then $x \in f(s) = X \sim F_s$, and if $u = z$, then $z \in g(s) = X \sim K_s$; hence we have a contradiction in both cases.

By Fleissner's theorem in [2], Y is collectionwise Hausdorff. Consequently, there are $H_x \in \mathcal{H}$, $x \in X$, and for each $x \in X$, a map g_x from H_x to the neighborhoods of x in X such that for any two distinct points x and y of X, $U(x, g_x) \cap U(y, g_y) = \varnothing$. For each $s \in {}^{<\omega}\omega$, let $D_s = \{x \in X: s \in H_x\}$. Then $X = \bigcup_{s \in {}^{<\omega}\omega} D_s$, and for each $s \in S$, the subspace D_s of X is discrete because, for each $x \in D_s$ and for any $y \in D_s \sim \{x\}$, we have $\langle y, x, s\rangle \in U(y, g_y) \sim U(x, g_x)$ and hence y cannot belong to the neighborhood $g_x(s)$ of x. $\square$

REMARKS. 1°. Note that the construction used above can be used (with a slight change in the proof of normality of Y) to show that, under $V = L$, a space of character $\leqslant \omega_1$ is σ-discrete provided that each subset of the space is $\mathcal{G}$-analytic (i.e., can be obtained by the A-operation from a family of open sets).

2°. In a σ-discrete space, every subset is a Borel-set (more precisely, the countable union of sets, each of which is the intersection of an open set with a closed set). In spite of Theorem 2.10, the following problem remains open: Is it true under $V = L$ that a space of character $\leqslant \omega_1$ has to be σ-discrete if each subset of the space is a Borel-set? The one-point compactification of an uncountable discrete space provides an example of a space in which each subset is Borel while there are nonanalytic subsets (note that if $x \in X$ is such that $X \sim \{x\}$ is analytic, then $\{x\}$ is a G_δ-set).

3. Some related results. An interesting problem motivated by the previous results is whether a metrizable space has to be σ-discrete under $V = L$ provided each subset of the space is projective. We have been unable to solve that problem even for the first level of projectivity (Σ^1_2) beyond analyticity. A related problem is whether a metrizable space has to be σ-discrete under $V = L$ in case each subset of the space belongs to the σ-algebra generated by the analytic subsets of the space. A positive solution to the latter problem is contained in the following results.

Let us call a subset A of a topological space X *strongly analytic* if there are open subsets G_s of X, for $s \in {}^{<\omega}\omega$, such that

$$A = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} G_{\phi|n} = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} \overline{G_{\phi|n}}.$$

Each strongly analytic set is both analytic and $\mathcal{G}$-analytic. On the other hand, it is easy to see that in a perfectly normal space, every analytic set is strongly analytic.

PROPOSITION 3.1 ($V = L$). *If X is a space of character $\leqslant \omega_1$ such that each subset of X belongs to the σ-algebra generated by the strongly analytic subsets of X, then X is σ-discrete.*

PROOF. Assume that X satisfies the conditions stated in the proposition. Denote by $\mathcal{P}(X)$ (by $\mathcal{S}$) the family consisting of all (strongly analytic) subsets of X. For any family $\mathcal{L}$ of subsets of X, denote by $\mathcal{L}_\sigma$ (by $\mathcal{L}_\delta$) the family consisting of all countable

unions (intersections) of sets from $\mathfrak{L}$, and let ${}^{\sim}\mathfrak{L} = \{X \sim L : L \in \mathfrak{L}\}$. Let $\mathfrak{S}_0 = \mathfrak{S} \cup {}^{\sim}\mathfrak{S}$, and define $\mathfrak{S}_\alpha$, for $\alpha < \omega_1$, recursively by the formula $\mathfrak{S}_\alpha = \mathcal{C}_\alpha \cup {}^{\sim}\mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = (\bigcup_{\beta<\alpha} \mathfrak{S}_\beta)_\sigma \cup (\bigcup_{\beta<\alpha} \mathfrak{S}_\beta)_\delta$. Then $\bigcup_{\alpha<\omega_1} \mathfrak{S}_\alpha$ is the σ-algebra generated by $\mathfrak{S}$; consequently, $\bigcup_{\alpha<\omega_1} \mathfrak{S}_\alpha = \mathcal{P}(X)$. It follows by $V = L$ (or just CH) and Theorem 9 of [3] that there exists $\rho < \omega_1$ such that $\mathfrak{S}_\rho = \mathcal{P}(X)$.

The remainder of this proof is carried out in two stages. First we use the topology on $X \times (\{\omega\} \cup {}^{<\omega}\omega)$ defined by Hansell in the proof of the theorem in [5] to obtain a space Y such that for each $S \in \mathfrak{S}$, the sets $S \times \{\omega\}$ and $(X \sim S) \times \{\omega\}$ can be separated by open subsets of Y. Then we define a set $I \subset {}^{\rho+1}\omega$ and a topology on $Y \times (\{\omega\} \cup I)$ to obtain a space Z such that for each $S \in \mathfrak{S}_\rho$, the sets $S \times \{\omega\} \times \{\omega\}$ and $(X \sim S) \times \{\omega\} \times \{\omega\}$ can be separated by open subsets of Z; since $\mathfrak{S}_\rho = \mathcal{P}(X)$, the space Z turns out to be normal and, by Fleissner's theorem, collectionwise Hausdorff; finally, collectionwise Hausdorffness of Z implies that X is σ-discrete.

Let $Y = X \times (\{\omega\} \cup {}^{<\omega}\omega)$ and topologize Y as follows. Make all the points of $Y \sim X \times \{\omega\}$ isolated. Let the filter $\mathcal{H}$ on ${}^{<\omega}\omega$ be defined as in the proof of Theorem 2.10. A neighborhood-base for a point $\langle x, \omega\rangle \in X \times \{\omega\}$ consists of all sets of the form

$$U(x, g) = \{\langle x, \omega\rangle\} \cup \bigcup_{s \in H} (g(s) \times \{s\}),$$

where $H \in \mathcal{H}$ and g is a map with domain H and range a set of neighborhoods of x in X. As $\chi(X) \leqslant \omega_1$, it is easily seen that $\chi(Y) \leqslant \omega_1$. Let S be a strongly analytic subset of X. We show that $S \times \{\omega\}$ and $(X \sim S) \times \{\omega\}$ can be separated by open subsets of Y. Let G_s, $s \in {}^{<\omega}\omega$, be open subsets of X such that $S = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} G_{\phi|n} = \bigcup_{\phi \in {}^{\omega}\omega} \bigcap_{n \in \omega} \overline{G}_{\phi|n}$. The proof of Lemma 2 of [5] shows that we can choose the sets G_s so that for all $\phi, \psi \in {}^{\omega}\omega$ and $n \in \omega$, $G_{\phi|n+1} \subset G_{\phi|n}$ and, if $\phi \leqslant \psi$, then $G_{\phi|n} \subset G_{\psi|n}$. For each $x \in X$, let $H_x = \{s \in {}^{<\omega}\omega : x \in G_s\}$ and $J_x = \{s \in {}^{<\omega}\omega : x \notin \overline{G}_s\}$; note that if $x \in S$, then $H_x \in \mathcal{H}$ and if $x \in X \sim S$, then $J_x \in \mathcal{H}$. For each $s \in {}^{<\omega}\omega$, let $g(s) = G_s$ and $f(s) = X \sim \overline{G}_s$. Let $V = \bigcup_{x \in S} U(x, g \mid H_x)$ and $W = \bigcup_{x \in X \sim S} U(x, f \mid J_x)$. Then V and W are open subsets of Y containing $S \times \{\omega\}$, respectively. It is easy to see that $V \cap W = \varnothing$.

Let I be the set of all step-functions in ${}^{\rho+1}\omega$ (i.e., $f \in I$ iff $f \in {}^{\rho+1}\omega$ and there are ordinals $0 = \alpha_1 < \cdots < \alpha_n = \rho + 1$, for some $n \in \omega$, such that f is constant on each of the intervals $[\alpha_i, \alpha_{i+1}[\ \subset \rho + 1)$. Note that $|I| = \omega$. We use induction on α to define a filter-base $\bigcup_{\alpha \leqslant \rho} \mathcal{E}_\alpha$ on I. Let

$$\mathcal{E}_0 = \{\{f \in I : f(0) \geqslant n\} : n < \omega\}.$$

If $\mathcal{E}_\beta$ has been defined for each $\beta < \alpha \leqslant \rho$, then $\mathcal{E}_\alpha$ consists of all sets of the form $\bigcup_{k \geqslant n} \{s \in E_k : s(\alpha) = k\}$, where $n \in \omega$ and $E_k \in \bigcup_{\beta<\alpha} \mathcal{E}_\beta$ for each $k \geqslant n$. Note that for all $\beta < \alpha \leqslant \rho$, $\mathcal{E}_\beta \subset \mathcal{E}_\alpha$; consequently, $\bigcup_{\alpha \leqslant \rho} \mathcal{E}_\alpha = \mathcal{E}_\rho$. Each of the families $\mathcal{E}_\alpha$ is closed under finite intersections: this holds trivially for $\alpha = 0$; if it holds for each $\alpha < \gamma$, and if $n, m \in \omega$ and $E_k, H_i \in \bigcup_{\alpha<\gamma} \mathcal{E}_\alpha$ for $k \geqslant n$ and $i \geqslant m$, then

$$\bigcup_{k \geqslant n} \{s \in E_k : s(\gamma) = k\} \cap \bigcup_{i \geqslant m} \{s \in H_i : s(\gamma) = i\}$$
$$= \bigcup_{j \geqslant \max(n,m)} \{s \in E_j \cap H_j : s(\gamma) = j\} \in \mathcal{E}_\gamma$$

so that $\mathcal{E}_\gamma$ is closed under finite intersections. In particular, $\mathcal{E}_\rho$ is closed under finite intersections. Each $\mathcal{E}_\alpha$ consists of nonempty sets. To see this, we use induction on α to show that for each $\alpha \leqslant \rho$, and for all $f \in I$ and $E \in \mathcal{E}_\alpha$, there exists $g \in E$ such that $g \mid \{\beta\colon \alpha < \beta \leqslant \rho\} = f \mid \{\beta\colon \alpha < \beta \leqslant \rho\}$. The statement holds trivially for $\alpha = 0$; assume it holds for each $\alpha < \gamma$. Let $f \in I$ and $E \in \mathcal{E}_\gamma$. Let $n \in \omega$ and $E_k \in \bigcup_{\alpha<\gamma} \mathcal{E}_\alpha$, for $k \geqslant n$, be such that $E = \bigcup_{k \geqslant n}\{s \in E_k\colon s(\gamma) = k\}$. Define $f' \in {}^{\rho+1}\omega$ by setting $f'(\gamma) = n$ and $f'(\alpha) = f(\alpha)$ for each $\alpha \neq \gamma$; note that $f' \in I$. Let $\delta < \gamma$ be such that $E_n \in \mathcal{E}_\delta$. By the induction assumption, there exists $g \in E_n$ such that

$$g \mid \{\beta\colon \delta < \beta \leqslant \rho\} = f' \mid \{\beta\colon \delta \leqslant \beta \leqslant \rho\}.$$

Clearly, $g \in E$ and $g \mid \{\beta\colon \gamma < \beta \leqslant \rho\} = f \mid \{\beta\colon \gamma < \beta \leqslant \rho\}$; this completes the induction. It follows that $\mathcal{E}_\rho$ consists of nonempty sets; hence $\mathcal{E}_\rho$ is a filter-base.

Let $Z = Y \times (\{\omega\} \cup I)$. Topologize Z as follows. Make all the points of $Z \sim X \times \{\omega\} \times \{\omega\}$ isolated. A neighborhood-base for a point $\langle x, \omega, \omega\rangle \in X \times \{\omega\} \times \{\omega\}$ consists of all sets of the form

$$V(x, g) = \{\langle x, \omega, \omega\rangle\} \cup \bigcup_{f \in E} (\varphi(f) \times \{f\}),$$

where $E \in \mathcal{E}_\rho$ and φ is a map with domain E and range a set of neighborhoods of $\langle x, \omega\rangle$ in Y. Using that $\chi(Y) \leqslant \omega_1$, we easily see that $\chi(Z) \leqslant \omega_1$. Note that $X \times \{\omega\} \times \{\omega\}$ is a closed and discrete subset of Z with discrete complement; hence, to show that Z is normal, it suffices to show that for each $S \in \mathcal{P}(X)$, the sets $S \times \{\omega\} \times \{\omega\}$ and $(X \sim S) \times \{\omega\} \times \{\omega\}$ can be separated by open subsets of Z. Call an open subset G of Z α-open, where $\alpha \leqslant \rho$, provided that for each $\langle x, \omega, \omega\rangle \in G \cap X \times \{\omega\} \times \{\omega\}$, there exists a neighborhood $V(x, \varphi)$ of $\langle x, \omega, \omega\rangle$ contained in G such that $\operatorname{dom}(\varphi) \in \mathcal{E}_\alpha$. For each $\alpha \leqslant \rho$, denote by $\mathcal{R}_\alpha$ the family consisting of all sets $R \subset X$ such that the sets $R \times \{\omega\} \times \{\omega\}$ and $(X \sim R) \times \{\omega\} \times \{\omega\}$ can be separated by α-open subsets of Z; note that $\mathcal{R}_\alpha$ is closed under complementation, and because of the corresponding property of $\mathcal{E}_\alpha$, $\mathcal{R}_\alpha$ is closed under finite intersections. We use induction on α to show that, for each $\alpha \leqslant \rho$, $\mathcal{S}_\alpha \subset \mathcal{R}_\alpha$. For each $S \in \mathcal{S}$, the sets $S \times \{\omega\}$ and $(X \sim S) \times \{\omega\}$ can be separated by open sets in Y; from this it follows that $\mathcal{S}_0 \subset \mathcal{R}_0$. Let α be such that $0 < \alpha \leqslant \rho$ and, for each $\beta < \alpha$, $\mathcal{S}_\beta \subset \mathcal{R}_\beta$. To show that $\mathcal{S}_\alpha \subset \mathcal{R}_\alpha$, recall that $\mathcal{S}_\alpha = \mathcal{C}_\alpha \cup {}^\sim\mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = (\bigcup_{\beta<\alpha} \mathcal{S}_\beta)_\sigma \cup (\bigcup_{\beta<\alpha} \mathcal{S}_\beta)_\delta$; since $\mathcal{R}_\beta$ is an algebra of sets for each $\beta \leqslant \alpha$, the inclusion $\mathcal{S}_\alpha \subset \mathcal{R}_\alpha$ follows once we show that for any decreasing sequence $\langle S_n\rangle_{n<\omega}$ of sets from $\bigcup_{\beta<\alpha} \mathcal{S}_\beta$, the set $\bigcap_{n<\omega} S_n$ belongs to $\mathcal{R}_\alpha$. Let S_n, $n < \omega$, be members of $\bigcup_{\beta<\alpha} \mathcal{S}_\beta$ such that for each $n < \omega$, $S_{n+1} \subset S_n$. For each $n < \omega$, $S_n \in \bigcup_{\beta<\alpha} \mathcal{R}_\beta$ and hence there are maps $\varphi_{x,n}$, with domain a member of $\bigcup_{\beta<\alpha} \mathcal{E}_\beta$ and range a set of neighborhoods of $\langle x, \omega\rangle$ in Y, for each $x \in X$, such that for all $x \in S_n$ and $y \in X \sim S_n$, $V(x, \varphi_{x,n}) \cap V(y, \varphi_{y,n}) = \varnothing$. Define the maps φ_x, $x \in X$, as follows. Let $x \in X$. If $x \in \bigcap_{n<\omega} S_n$, set $n(x) = 0$ and if $x \in X \sim \bigcap_{n<\omega} S_n$, let $n(x) < \omega$ be such that $x \notin S_{n(x)}$. Let $E_x = \bigcup_{k \geqslant n(x)} \{f \in \operatorname{dom}(\varphi_{x,k})\colon f(\alpha) = k\}$, and note that $E_x \in \mathcal{E}_\alpha$. We make E_x the domain of φ_x, and for each $f \in E_x$, we let $\varphi_x(f)$ be the neighborhood $\varphi_{x,f(\alpha)}(f)$ of $\langle x, \omega\rangle$ in Y. Clearly, $\bigcup\{V(x, \varphi_x)\colon x \in \bigcap_{n<\omega} S_n\}$ and

$\bigcup\{V(x, \varphi_x): x \in X \sim \bigcap_{n<\omega} S_n\}$ are α-open subsets of Z containing $(\bigcap_{n<\omega} S_n) \times \{\omega\} \times \{\omega\}$ and $(X \sim \bigcap_{n<\omega} S_n) \times \{\omega\} \times \{\omega\}$, respectively. To see that these sets are disjoint, assume on the contrary that there are $x \in \bigcap_{n<\omega} S_n$ and $y \in X \sim \bigcap_{n<\omega} S_n$ such that $V(x, \varphi_x) \cap V(y, \varphi_y) \neq \varnothing$. Then there exists $f \in \mathrm{dom}(\varphi_x) \cap \mathrm{dom}(\varphi_y)$ such that $\varphi_x(f) \cap \varphi_y(f) \neq \varnothing$. Let $k = f(\alpha)$. Since $f \in \mathrm{dom}(\varphi_y)$, we have $k \geqslant n(y)$ and hence $y \in X \sim S_k$. It follows that $V(x, \varphi_{x,k}) \cap V(y, \varphi_{y,k}) = \varnothing$ and further, since $f \in \mathrm{dom}(\varphi_{x,k}) \cap \mathrm{dom}(\varphi_{y,k})$, that $\varphi_{x,k}(f) \cap \varphi_{y,k}(f) = \varnothing$; however, this is a contradiction, since $\varphi_{x,k}(f) = \varphi_x(f)$ and $\varphi_{y,k}(f) = \varphi_y(f)$. We have shown that $\bigcap_{n<\omega} S_n \in \mathfrak{R}_\alpha$. This completes the proof of the inductive step. Since $\mathfrak{S}_\rho \subset \mathfrak{R}_\rho$ and $\mathfrak{S}_\rho = \mathfrak{P}(X)$, we have proved that Z is normal.

By the theorem in [2], Z is collectionwise Hausdorff. Consequently, we can find mutually disjoint basic neighborhoods $V(x, \varphi_x)$ for the points $\langle x, \omega, \omega\rangle$ of the closed discrete set $X \times \{\omega\} \times \{\omega\}$. For each $x \in X$, let $f_x \in \mathrm{dom}(\varphi_x)$, and choose a basic neighborhood $U(x, g_x)$ contained in the neighborhood $\varphi_x(f_x)$ of $\langle x, \omega\rangle$ in Y, and let $s_x \in \mathrm{dom}(g_x)$; note that $g_x(s_x)$ is a neighborhood of x in X. For all $f \in I$ and $s \in {}^{<\omega}\omega$, let $D_{f,s} = \{x \in X: f_x = f \text{ and } s_x = s\}$; note that $D_{f,s}$ is a discrete subspace of X since for all distinct x and y in $D_{f,s}$, $g_x(s_x) \cap g_y(s_y) = \varnothing$. Since $|I \times {}^{<\omega}\omega| = \omega$, it follows that X is σ-discrete. □

COROLLARY 3.2 ($V = L$). *If X is a perfectly normal space of character $\leqslant \omega_1$ such that each subset of X belongs to the σ-algebra generated by the analytic subsets of X, then X is F_σ-discrete.*

Since each metrizable space is perfectly normal and has character $\leqslant \omega_1$, Corollary 3.2 and Proposition 2.2 of [7], give the following result.

COROLLARY 3.3 ($V = L$). *If X is a σ-space such that each subset of X belongs to the σ-algebra generated by the analytic subsets of X, then X is F_σ-discrete.*

REFERENCES

1. A. V. Arhangel'skiĭ and S. J. Nedev, *Some remarks on semi-metrizable spaces and their subspaces*, C. R. Acad. Bulgare Sci. **31** (1978), no. 5, 499–500.
2. W. G. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. **46** (1974), 294–298.
3. D. H. Fremlin, R. W. Hansell and H. J. K. Junnila, *Borel functions of bounded class*, Trans. Amer. Math. Soc. (to appear).
4. J. Gerlits and J. Juhász, *On left-separated compact spaces*, Comment. Math. Univ. Carolinae **19** (1978), 53–62.
5. R. W. Hansell, *Some consequences of ($V = L$) in the theory of analytic sets*, Proc. Amer. Math. Soc. **80** (1980), 311–319.
6. H. J. K. Junnila, *Neighbornets*, Pacific J. Math. **76** (1978), 83–108.
7. _____, *On σ-spaces and pseudometrizable spaces*, Topology Proc. **4** (1979), 121–132.
8. G. M. Reed, *On normality and countable paracompactness*, Fund. Math. **110** (1980), 145–152.

DEPARTMENT OF MATHEMATICS, KOSSUTH UNIVERSITY, DEBRECEN PF. 12, 4010 HUNGARY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HALLITUSKATU 15, HELSINKI 10, FINLAND