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LEBESGUE SETS AND INSERTION OF A CONTINUOUS FUNCTION

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ABSTRACT. Necessary and sufficient conditions in terms of Lebesgue sets are presented for the following two insertion properties for real-valued functions defined on a topological space: (1) If $g \leq f$ there is a continuous function h such that $g \leq h \leq f$, and for each x for which $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. (2) If $g < f$ there is a continuous function h such that $g < h < f$.

1. Statement of results. All functions considered are real-valued. Let R (respectively, Q) denote the real (respectively, rational) numbers and write $g \leq f$ (respectively, $g < f$) in case $g(x) \leq f(x)$ (respectively, $g(x) < f(x)$) for each x in the space. For b in R the Lebesgue sets for a function f are defined by $L_b(f) = \{x : f(x) \leq b\}$ and $L^b(f) = \{x : f(x) \geq b\}$. Let $C^*(X)$ denote the lattice of continuous, bounded, and real-valued functions on X. The main results of this paper utilize Lebesgue sets to characterize certain insertion properties of a continuous function, and they are based on the following ([**2**, Theorem 3.5; **8**, Theorem 2.1]):

THEOREM 1. *If $g \leq f$ then there is a continuous function h such that $g \leq h \leq f$ if and only if for any rational numbers a and b such that $a < b$ the Lebesgue sets $L^b(g)$ and $L_a(f)$ are completely separated.*

The following is stated on p. 444 of [**11**].

THEOREM 2. *Let X be a topological space and let $L(X)$ and $U(X)$ be classes of bounded functions defined on X such that any constant function is in $L(X) \cap U(X)$ and such that if $g \in U(X)$, $f \in L(X)$, and $r \in R$ then $g \wedge r \in U(X)$ and $f \vee r \in L(X)$. The following are equivalent:*

(i) *If $f \in L(X)$, $g \in U(X)$, and $g \leq f$ there exists h in $C^*(X)$ such that $g \leq h \leq f$ and such that $g(x) < h(x) < f(x)$ for each x for which $g(x) < f(x)$.*
(ii) *If $f \in L(X)$, $g \in U(X)$, and $r \in R$ the Lebesgue sets $L_r(f)$ and $L^r(g)$ are zero sets in X.*
(iii) *If $f \in L(X)$ and $g \in U(X)$ then f (respectively, g) is the pointwise limit of and increasing (respectively, decreasing) sequence of continuous functions.*

In the situation where $U(X)$ and $L(X)$ are the classes of upper and lower semicontinuous functions, respectively, the equivalence of (i) and (ii) of the above theorem is due to Michael [**12**], the equivalence of (ii) and (iii) is due to Tong [**14**], and each of the conditions being equivalent to X is perfectly normal. If $U(X)$ and $L(X)$ are the classes of normal upper and normal lower semicontinuous functions,

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respectively, the equivalence of (i), (ii), (iii), and X is an Oz space is established in [**8**].

Necessary and sufficient conditions in order for a space to satisfy condition (i) of Theorem 2 for general classes of functions are considered in [**12**, **3**, **4**, **7**, and **10**]. Let $B(X)$ denote the Banach lattice of all bounded real-valued functions on a space X. If C is a sublattice of the power set of X to which $\varnothing$ and X belong, the smallest convex cone in $B(X)$ that contains the constant functions 1_D, $D \in C$, is denoted by $cn(C)$ and its closure by $\overline{cn}(C)$. The results of Blatter and Seever in [**3** and **4**] require that the classes $U(X)$ and $L(X)$ can be characterized as $\overline{cn}(A)$ and $\overline{cn}(B)$, respectively, for some sublattices A and B of the power set of X and that $A \subset B_\delta$ (= intersection of sequences in B) and $B \subset A_\sigma$ (= union of sequences in A). The necessary (as proved by Powderly [**13**]) and sufficient condition of [**7**] places restrictions on the function $f-g$. These limitations are avoided in Theorem 2.

A portion of the following result is stated on p. 478 of [**11**].

THEOREM 3. *Let $L(X)$ and $U(X)$ be classes of bounded real-valued functions on X such that $C^*(X) \subset L(X) \cap U(X)$. The following are equivalent:*

(i) *For $f \in L(X)$, $g \in U(X)$ and $g < f$ there exists $h \in C^*(X)$ such that $g < h < f$.*

(ii) *If $f \in L(X)$, $g \in U(X)$ and $g < f$ then for each r in Q there exist disjoint sets A_r and B_r such that $L_r(f)$ and $X - A_r$ are completely separated, $L^r(g)$ and $X - B_r$ are completely separated, and each of $\{X - (B_r \cup L_r(f)) \colon r \in Q\}$ and $\{X - (A_r \cup L^r(g)) \colon r \in Q\}$ covers X.*

(iii) *If $f \in L(X)$, $g \in U(X)$ and $g < f$ then for each rational number r there exist disjoint sets A_r and B_r such that $L_r(f)$ and $X - A_r$ are completely separated, $L^r(g)$ and $X - B_r$ are completely separated, and $\{X - (A_r \cup B_r) \colon r \in Q\}$ covers X.*

In the situation in which $L(X)$ and $U(X)$ are the lattices of lower and upper semicontinuous functions, respectively, results of Dowker [**5**] and Katětov [**6**] show that a space satisfies (i) of the above theorem if and only if X is normal and countably paracompact. Other specific cases are given in [**10**, Theorem 4.2]. Necessary and sufficient conditions for a space to satisfy (i) of Theorem 3 for general classes of functions are given in [**3**, **4**, and **7**] but these results have restrictions analogous to those mentioned in the discussion following Theorem 2.

It is noted that the bounded condition placed on the functions in Theorems 2 and 3 causes no loss in generality if the properties that define the classes $L(X)$ and $U(X)$ are preserved under an order preserving homeomorphism from R onto a finite interval.

2. Proofs of results. The following lemma is used in combination with Theorem 1 in proving Theorems 2 and 3. The argument is adapted from a technique used in the proof of Theorem 3.3 in [**3**] and is included here for completeness.

LEMMA 1. *Let f, g and k be bounded functions such that $g \leq k \leq f$ and $k \in C^*(X)$. If there exist sequences $\{a_n\}$ and $\{b_n\}$ in $C^*(X)$ such that $g \leq a_n$ and $b_n \leq f$ for all n, $\inf_n a_n(x) < f(x)$ and $\sup_n b_n(x) > g(x)$ for each x for which $g(x) < f(x)$, then there exists h in $C^*(X)$ such that $g \leq h \leq f$ and for each x for which $g(x) < f(x)$ then $g(x) < h(x) < f(x)$.*

PROOF. Using the notation of the lemma, set

$$h = \sum_{n=1}^{\infty} 2^{-n-1}(a_n \wedge k + b_n \wedge k).$$

Since $g \le a_n \wedge k$, $\sum_{n=1}^{\infty} 2^{-n} g \le \sum_{n=1}^{\infty} 2^{-n}(a_n \wedge k)$, or $g \le \sum_{n=1}^{\infty} 2^{-n}(a_n \wedge k)$. Similarly, from $f \ge a_n \wedge k$ it follows that $f \ge \sum_{n=1}^{\infty} 2^{-n}(a_n \wedge k)$. Thus $g \le \sum_{n=1}^{\infty} 2^{-n}(a_n \wedge k) \le f$. In the same fashion show that $g \le \sum_{n=1}^{\infty} 2^{-n}(k \vee b_n) \le f$. From the definition of h it follows that $g \le h \le f$. Let x be such that $g(x) < f(x)$. Choose N so that $a_N(x) < f(x)$. Then $\sum_{n=1}^{\infty} 2^{-n}(a_n \wedge k)(x) = \sum_{n \ne N} 2^{-n}(a_n \wedge k)(x) + 2^{-N}(a_N \wedge k)(x) < \sum_{n \ne N} 2^{-n}(a_n \wedge k)(x) + 2^{-N} f(x) \le \sum_{n=1}^{\infty} 2^{-n} f(x) = f(x)$. Similarly, show that $g(x) < \sum_{n=1}^{\infty} 2^{-n}(b_n \vee k)(x)$. Hence $g(x) < h(x) < f(x)$ whenever $g(x) < f(x)$.

That condition (ii) of Theorem 2 implies (i) is a consequence of Proposition 6.1 of [**3**]. A proof is given here that uses the above lemma since this approach seems considerably more direct.

PROOF OF THEOREM 2. (i)$\Rightarrow$(ii) If $g \in U(X)$ and $r \in R$ then by hypothesis $g \wedge r \in U(X)$ and $r \in L(X)$. By (i) there is a continuous function h such that $g \wedge r \le h \le r$ and if $g(x) < r$ then $g(x) < h(x) < r$; $L^r(g)$ is a zero set since $L^r(g) = \{x\colon h(x) = r\}$. Similarly show that $L_r(f)$ is a zero set.

(ii)$\Rightarrow$(iii) If f is a lower semicontinuous function defined on a perfectly normal space, Tong's proof [**14**] that f is a pointwise limit of an increasing sequence of continuous functions is based on the Lebesgue set $L_r(f)$ being a zero set; with trivial modification his proof yields this implication.

(iii)$\Rightarrow$(i) Let $g \in U(X)$ and $f \in L(X)$ with $g \le f$. If $\{f_n\}$ is an increasing sequence of continuous functions whose pointwise limit is f then for any real number r the Lebesgue set $L_r(f)$ equals the intersection of the sequence $\{L_{r+1/n}(f_n)\}$ of zero sets, and thus $L_r(f)$ is a zero set. Similarly, use a decreasing sequence $\{g_n\}$ of continuous functions whose pointwise limit is g to show that each $L^r(g)$ is a zero set. For any rational numbers $a < b$, $L^b(g)$ and $L_a(f)$ are disjoint zero sets and hence are completely separated; by Theorem 1 there is a continuous function k such that $g \le k \le f$. Since the sequences $\{g_n\}$ and $\{f_n\}$ satisfy the conditions of Lemma 1, it follows that there exists h in $C^*(X)$ such that $g \le h \le f$ and whenever $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. This concludes the proof of Theorem 2.

If k maps a space X into R, call k *regular lower semicontinuous* (respectively, *regular upper semicontinuous*) if for each real number r the Lebesgue set $L_r(k)$ (respectively, $L^r(k)$) is a regular G_δ subset of X. (These functions were considered in [**9**].) Let $L(X)$ (respectively, $U(X)$) denote the class of bounded regular lower (respectively, upper) semicontinuous functions. The following is an immediate corollary of Theorem 2. If $g \in U(X)$, $f \in L(X)$, and $g \le f$ there is h in $C^*(X)$ such that $g \le h \le f$ and such that $g(x) < h(x) < f(x)$ whenever $g(x) < f(x)$ if and only if each regular G_δ subset of X is a zero set. (If X is an Oz space [**1**] or if X is almost normal then each regular G_δ subset of X is a zero set.)

PROOF OF THEOREM 3. (i)$\Rightarrow$(iii) If $f \in L(X)$, $g \in U(X)$ and $g < f$ then by (i) there exists h in $C^*(X)$ such that $g < h < f$. Since $C^*(X) \subset L(X) \cap U(X)$ we may use hypothesis (i) again to show there exist h_1 and h_2 in $C^*(X)$ such that $g < h_1 < h < h_2 < f$. For each r in Q let $A_r = \{x\colon h_2(x) < r\}$ and $B_r = \{x\colon h_1(x) > r\}$. In order to see that $L_r(f)$ and $X - A_r$ are completely separated, use (i) to choose k

in $C^*(X)$ such that $h_2 < k < f$. Since $X - A_r$ and $L_r(k)$ are disjoint zero sets and $L_r(k) \supset L_r(f)$ it follows that $L_r(f)$ and $X - A_r$ are completely separated. Similarly, $L^r(g)$ and $X - B_r$ are completely separated. The sets $X - (A_r \cup B_r)$, $r \in Q$, cover X since $h_1 < h_2$. Thus (iii) is satisfied.

That (iii) implies (ii) is manifest; the argument to show (ii) implies (i) follows: Let $g \in U(X)$, $f \in L(X)$ and suppose that $-M < g < f < M$. It follows from (ii) that for any rational number r that $L_r(f)$ and $L^r(g)$ are completely separated; in particular for any rationals $a < b$ then $L_a(f)$ and $L^b(g)$ are completely separated. By Theorem 1 there is k in $C^*(X)$ such that $g \leq k \leq f$. For each rational number r choose sets A_r and B_r satisfying the conditions of (ii), and then choose a_r and b_r in $C^*(X)$ such that $-M \leq a_r \leq r$, $a_r = -M$ on $L_r(f)$, $a_r = r$ on $X - A_r$, $r \leq b_r \leq M$, $b_r = M$ on $L^r(g)$, and $b_r = r$ on $X - B_r$. If $x \in L_r(f)$ then $a_r(x) = -M \leq f(x)$ and if x is not in $L_r(f)$ then $f(x) > r \geq a_r(x)$; thus $a_r \leq f$. Let $x \in X$ and choose $s \in Q$ such that x is $X - (A_s \cup L^s(g))$; since $x \in X - L^s(g)$ then $g(x) < s$ and since $x \in X - A_s$ then $a_s(x) = s$. Thus $\sup_r a_r(x) \geq a_s(x) > g(x)$. Similarly, show that $g \leq b_r$ for each r and $\inf_r b_r(x) < f(x)$ for each x. By Lemma 1 there exists h in $C^*(X)$ such that $g < h < f$. This concludes the proof of Theorem 3.

As one application of Theorem 3 consider the result of Dowker and Katětov mentioned above. Suppose that X is normal and countably paracompact, g is upper semicontinuous, f is lower semicontinuous and $g < f$. Since $\{X - (L_r(f) \cup L^r(g))\colon r \in Q\}$ is a countable open cover of X there is a cover $\{F_r\colon r \in Q\}$ of X such that F_r is closed and $F_r \subset X - (L_r(f) \cup L^r(g))$ for each r. Since F_r and $L_r(f)$ are completely separated choose k_r in $C^*(X)$ such that $k_r = 0$ on $L_r(f)$, $k_r = 1$ on F_r, and let $A_r = \{x\colon k_r(x) < \frac{1}{2}\}$. Thus $L_r(f)$ and $X - A_r$ are completely separated. Similarly, define B_r so that $L^r(g)$ and $X - B_r$ are completely separated. Since $F_r \subset X - (A_r \cup B_r)$, $\{X - (A_r \cup B_r)\colon r \in Q\}$ covers X. By Theorem 3 there is h in $C^*(X)$ such that $g < h < f$.

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