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UNIFORM σ-ADDITIVITY IN SPACES OF BOCHNER OR PETTIS INTEGRABLE FUNCTIONS OVER A LOCALLY COMPACT GROUP

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ABSTRACT. If G is an abelian locally compact group with Haar measure μ, E is a Banach space and $K \subset L^1_E(\mu)$, we give necessary and sufficient conditions for the set $\{\int_{(\cdot)} |f|\,d\mu;\ f \in K\}$ to be uniformly σ-additive in terms of uniform convergence on K, for the topology $\sigma(L^1_E, L^\infty_{E'})$ of convolution and translation operators. In case $E = R$, this gives a new characterization of relatively weakly compact sets $K \subset L^1$.

1. Introduction. In this paper we consider the space L^1_E of Bochner integrable functions and the space $\mathfrak{L}^1_E$ of Pettis integrable functions over an abelian locally compact group G endowed with a Haar measure μ, and we give a characterization of uniform σ-additivity in terms of uniform convergence—in the topology $\sigma' = \sigma(L^1_E, L^\infty_{E'})$, respectively in the weak topology of $\mathfrak{L}^1_E$—of convolution operators and translation operators. If $E = R$, this yields a new characterization of relative weak compactness in L^1.

The convolution of Bochner integrable functions has been studied in **[2]** and has been extended in **[4]** for Pettis integrable functions.

Similar results have been obtained in a previous paper **[3]**, where we give a characterization of uniform σ-additivity in the spaces L^1_E and $\mathfrak{L}^1_E$ over a measure space (X, Σ, μ), in terms of uniform convergence, in the σ'-topology or in the weak topology, of conditional expectations.

2. Uniform σ-additivity in the Lebesgue space L^1_E. Let G be an abelian locally compact additive group endowed with a Haar measure μ; let E be a Banach space and L^1_E be the space of Bochner μ-integrable functions $f\colon G \to E$. For each relatively compact neighborhood V of 0 in G, we choose a function u_V on G which is positive, bounded, symmetric (i.e. $u_V(-t) = u_V(t)$), vanishes outside V and $\int u_V\,d\mu = 1$. If $\mathcal{V}$ is a base of relatively compact neighborhoods of 0 in G, we call $(u_V)_{V\in\mathcal{V}}$ an approximate unit. We denote by $u_V * f$ the convolution: $u_V * f(t) = \int u_V(t-s)f(s)\,ds$, for $t \in G$. For $h \in G$ we denote by T^h the translation operator, defined by $(T^hf)(t) = f(t+h)$, for $t \in G$. If $f \in L^1_E$, we denote by $f\mu$ the measure defined for any Borel set $A \subset G$ by $(f\mu)(A) = \int_A f\,d\mu$. Finally, we denote by σ' the topology $\sigma(L^1_E, L^\infty_{E'})$ on

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$L^1_{E'}$. If E' has the Radon-Nikodym property, then the σ'-topology is the weak topology of $L^1_{E'}$.

THEOREM 1. *Let $K \subset L^1_E$ be a set.*

I. *The set $|K|\mu = \{|f|\mu;\ f \in K\}$ is uniformly σ-additive, if and only if the following conditions are satisfied*:

(a) *K is bounded in L^1_E*;

(b) *For every countable subset $K_0 \subset K$, there exists a decreasing sequence (V_n) of neighborhoods of* 0 *in G, such that either*

(b$_1$) $\lim_n u_{V_n} * f = f$ *in L^1_E for the σ'-topology, uniformly for $f \in K_0$*; *or*

(b$_2$) $\lim_{h \in V_n, n\to\infty} T^h f = f$, *in L^1_E, for the σ'-topology, uniformly for $f \in K_0$*;

(c) $\lim_C \chi_C f = f$, *strongly in L^1_E, uniformly for $f \in K$, where the limit is taken along the increasing net of all compact subsets of G.*

(*Condition* (c) *is superfluous if all functions of K vanish outside a common compact set*; *in particular, if G is compact.*)

II. *If $|K|\mu$ is uniformly σ-additive, then*

(b$'_1$) $\lim_V u_V * f = f$ *and*

(b$'_2$) $\lim_{h\to 0} T^h f = f$,

in L^1_E, for the σ'-topology, uniformly for $f \in K$.

PROOF. Assume first conditions (a), (b$_1$) and (c) satisfied and prove that $|K|\mu$ is uniformly σ-additive. Let $C \subset G$ be a compact set and $\phi \in L^1 \cap L^\infty$ a function with compact support V.

(A) The set $\phi * (\chi_C K) = \{\phi * (\chi_C f); f \in K\}$ is bounded in L^1_E. In fact, if $f \in K$, then

$$\|\phi * (\chi_C f)\|_1 \leqslant \|\phi\|_1 \|\chi_C f\|_1 \leqslant M\|\phi\|_1,$$

where $M = \sup\{\|f\|_1;\ f \in K\}$.

(B) The set $|\phi * (\chi_C K)|\mu$ is uniformly σ-additive. In fact the set $\phi * (\chi_C K)$ is bounded in L^∞_E:

$$\|\phi * (\chi_C f)\|_\infty \leqslant \|\phi\|_\infty \|\chi_C f\|_1 \leqslant M\|\phi\|_1, \quad \text{for } f \in K.$$

It follows that the set $|\phi * (\chi_C K)|\mu$ is uniformly absolutely μ-continuous. Since all the functions of $\phi * (\chi_C K)$ vanish outside the compact set $C + V$, the set $|\phi * (\chi_C K)|\mu$ is uniformly σ-additive.

(C) For every $g \in L^\infty_{E'}$ and $f \in K$ we have

$$\left|\int \langle \phi * (\chi_C f) - f, g\rangle\, d\mu\right| \leqslant \left|\int \langle \phi * (\chi_C f - f), g\rangle\, d\mu\right| + \left|\int \langle \phi * f - f, g\rangle\, d\mu\right|$$
$$\leqslant \|\phi\|_1 \|\chi_C f - f\|_1 \|g\|_\infty + \left|\int \langle \phi * f - f, g\rangle\, d\mu\right|.$$

From condition (c) we deduce that there is an increasing sequence (C_n) of compact sets such that $\lim_n \|\chi_{C_n} f - f\|_1 = 0$, uniformly for $f \in K$. Let $K_0 \subset K$ be a countable set. Taking above $C = C_n$ and $\phi = u_{V_n}$, where (V_n) is the sequence stated in condition (b), we deduce that

$$\lim_n u_{V_n} * (\chi_{C_n} f) = f, \quad \text{in } L^1_E$$

for the σ'-topology, uniformly for $f \in K_0$. Since, for each n, the set $|u_{V_n} * (\chi_{C_n} K_0)|\mu$ is bounded and uniformly σ-additive, from Lemma Ib in [3] we deduce that $|K_0|\mu$ is also uniformly σ-additive. Since K_0 was an arbitrary countable set in K, it follows that $|K|\mu$ is uniformly σ-additive. If conditions (a), (b_2) and (c) are satisfied, then $|K|\mu$ is again uniformly σ-additive, since by Proposition 12 in [4], condition (b_2) implies (b_1). We remark that in [4], the implication $b_2 \Rightarrow b_1$ is stated for the weak topology, but the same proof is valid for the σ'-topology.

Conversely, assume $|K|\mu$ is uniformly σ-additive.

(D) K is bounded in L^1_E. In fact, we can find a compact set $B \subset G$ such that $\int_{G-B} |f|\, d\mu \leq 1$ for all $f \in K$. Since $|K|\mu$ is uniformly absolutely μ-continuous, there is $\eta > 0$ such that if $\mu(A) \leq \eta$, then $\int_A |f|\, d\mu \leq 1$ for all $f \in K$. Since the Haar measure is diffuse, it has the Darboux property: there is a finite family of disjoint Borel sets $A_1, \ldots, A_n$ with union B, such that $\mu(A_i) \leq \eta$ for $i = 1, \ldots, n$. It follows that $\int_B |f|\, d\mu \leq n$ for all $f \in K$, hence $\int |f|\, d\mu \leq n + 1$ for all $f \in K$; consequently K is bounded.

(E) Since $|K|\mu$ is uniformly σ-additive, for every $\varepsilon > 0$ there is a compact set $C \subset G$ such that

$$\int_{G-C} |f|\, d\mu < \varepsilon, \quad \text{for all } f \in K,$$

that is

$$\int |\chi_C f - f|\, d\mu < \varepsilon, \quad \text{for all } f \in K$$

and condition (c) follows.

(F) Let $f \in L^1_E$, $g \in L^\infty_{E'}$, $\lambda > 0$, $h \in G$, and C be an integrable subset. Then

$$\left|\int \langle T^h f - f, g \rangle\, d\mu\right| \leq 2\|g\|_\infty \int_{G-C} |f|\, d\mu + 2\|g\|_\infty \int_{\{|f| > \lambda\}} |f|\, d\mu$$
$$+\lambda \|T^{-h}(\chi_C g) - \chi_C g\|_1 + \lambda \|g\|_\infty \|\chi_C - \chi_{C-h}\|_1.$$

In fact

$$\left|\int \langle T^h f - f, g \rangle\, d\mu\right| \leq \left|\int \langle T^h(f\chi_{G-C}) - f\chi_{G-C}, g \rangle\, d\mu\right|$$
$$+ \left|\int \langle T^h(f\chi_C) - f\chi_C, g \rangle\, d\mu\right|.$$

The first term can be written

$$\left|\int \langle T^h(f\chi_{G-C}) - f\chi_{G-C}, g \rangle\, d\mu\right| = \left|\int \langle f\chi_{G-C}, T^{-h}g - g \rangle\, d\mu\right|$$
$$\leq 2\|g\|_\infty \int_{G-C} |f|\, d\mu.$$

For the second term we have $T^h(f\chi_C) = \chi_{C-h}T^h(f\chi_C)$, hence

$$\left|\int \langle T^h(f\chi_C) - f\chi_C, g\rangle\, d\mu\right| = \left|\int \langle f\chi_C, T^{-h}(g\chi_{C-h}) - g\chi_C\rangle\, d\mu\right|$$

$$\leq \left|\int_{C\cap\{|f|>\lambda\}} \langle f, T^{-h}(g\chi_{C-h}) - g\chi_C\rangle\, d\mu\right|$$

$$+\left|\int_{C\cap\{|f|\leq\lambda\}}\right| \langle f, T^{-h}(g\chi_{C-h}) - g\chi_C\rangle\, d\mu$$

$$\leq 2\|g\|_\infty \int_{\{|f|>\lambda\}} |f|\, d\mu + \lambda \|T^{-h}(g\chi_{C-h}) - g\chi_C\|_1$$

$$\leq 2\|g\|_\infty \int_{\{|f|>\lambda\}} |f|\, d\mu + \lambda \|T^{-h}(g\chi_C) - g\chi_C\|_1$$

$$+\lambda \|T^{-h}g(\chi_{C-h} - \chi_C)\|_1$$

and this last term is smaller than $\lambda\|g\|_\infty \|\chi_{C-h} - \chi_C\|_1$.

(G) We can now prove conditions (b_2') and (b_1'). Let $g \in L^\infty_{E'}$, $g \neq 0$ and $\varepsilon > 0$. Take $C \subset G$ such that

$$\int_{G-C} |f|\, d\mu < \varepsilon/(8\|g\|_\infty), \quad \text{for all } f \in K.$$

Take also $\lambda > 0$ such that

$$\int_{\{|f|>\lambda\}} |f|\, d\mu < \varepsilon/(8\|g\|_\infty), \quad \text{for all } f \in K.$$

We can find a symmetric neighborhood V of 0 such that for all $h \in V$ we have

$$\|T^{-h}(\chi_C g) - \chi_C g\|_1 < \varepsilon/4,$$

and

$$\|\chi_C - \chi_{C-h}\|_1 = \|\chi_C - T^h\chi_C\|_1 < \varepsilon/(4\lambda\|g\|_\infty).$$

Then, for $h \in V$ and all $f \in K$ we have, from step (F),

$$\left|\int \langle T^h f, g\rangle\, d\mu\right| < \varepsilon;$$

that is $\lim_{h\to 0} T^h f = f$, in L^1_E for the σ'-topology, uniformly for $f \in K$. This proves condition (b_2'); and condition (b_1') follows from Proposition 12 in [**3**].

(H) To prove condition (b_2), let $K_0 \subset K$ be a countable subset. The proof of condition (b_1) is the same as in step (G).

Let R_0 be a countable ring of relatively compact Borel subsets of G, such that any function of K_0 is the limit μ-a.e. and in L^1_E of step functions over R_0.

Since for each $f \in L^1_E$ we have $\lim_{h\to 0} T^h f = f$, strongly in L^1_E, we can find a decreasing sequence (V_n) of symmetric neighborhoods of 0, such that $\lim_{h\in V_n, n\to\infty} T^h\chi_A = \chi_A$, strongly in L^1 for every $A \in R_0$. Next, we choose arbitrarily a sequence (h_n) such that $h_{2n-1} = -h_{2n} \in V_n$ for every n. Then $\lim_n T^{h_n}\chi_A = \chi_A$ strongly in L^1 for every $A \in R_0$. Consider the group $\Gamma \subset G$ generated by the sequence (h_n). Then the set L of linear combinations of functions of the form $(T^{\alpha_1}\chi_{A_1})(T^{\alpha_2}\chi_{A_2}) \cdots (T^{\alpha_k}\chi_{A_k})$ with $\alpha_1,\ldots,\alpha_k \in \Gamma$ and $A_1,\ldots,A_k \in R_0$, is an algebra of μ-integrable functions, invariant with respect to T^α for any $\alpha \in \Gamma$. Moreover, $\lim_n T^{h_n}\chi = \chi$, in L^1, for all $\chi \in L$.

It is enough to check this for the functions of the form $\chi = (T^{\alpha_1}\chi_{A_1})(T^{\alpha_2}\chi_{A_2})$. We have $\lim_n T^{h_n}\chi = \chi$, μ-a.e. and since $|T^{h_n}\chi| \leq \chi_{V_1} + (A_1 \cup A_2)$, we can apply Lebesgue's dominated convergence theorem and deduce that $\lim_n T^{h_n}\chi = \chi$ in L^1.

Moreover, this last equality remains valid for χ in the closure of L in L^1, since $\sup_n \|T^{h_n}\| = 1$.

The class $\Lambda = \{A;\ \chi_A \in L\}$ is a ring containing R_0, and the class $\{\chi_A;\ A \in \Lambda\}$ is invariant with respect to T^α for all $\alpha \in \Gamma$. All functions of L vanish μ-a.e. outside a σ-finite set X_0.

The δ-ring Σ_0 generated by Λ is the completion of Λ for the semidistance $\rho(A, B) = \mu(A \Delta B) = \|\chi_A - \chi_B\|_1$, and can be obtained—modulo negligible sets—as closure in L^1 of the set of functions χ_A with $A \in \Lambda$. It follows that the class $\{\chi_A;\ A \in \Sigma_0\}$ is invariant with respect to T^α for all $\alpha \in \Gamma$, since the class $\{\chi_A;\ A \in \Lambda\}$ has this property, and since this property is preserved by passing to limits in L^1.

We deduce then that for any Banach space F, the space $L^1_F(X_0, \Sigma_0, \mu)$ is invariant with respect to T^α for all $\alpha \in \Gamma$, and that for every $f \in L^1_F(X_0, \mu)$ we have $\lim_n T^{h_n}f = f$, strongly in $L^1_F(X_0, \Sigma_0, \mu)$.

In fact this property is valid for all step functions, and $\sup_n \|T^{h_n}\| = 1$.

We are now ready to prove condition (b_2).

Let $g \in L^\infty_{E'}$, $g \neq 0$, and $\varepsilon > 0$. The conditional expectation $g' = E(g \mid \Sigma_0)$ is defined since the space (G, μ) is localizable (see [**1**]).

We can consider $g' \in L^\infty_{E'}(X_0, \Sigma_0, \mu)$. Since $K_0 \subset L^1_E(X_0, \Sigma_0, \mu)$ and $|K_0|\mu$ is uniformly σ-additive, there is a set $C \in \Sigma_0$ such that

$$\int_{G-C} |f|\, d\mu < \varepsilon/(8\|g\|_\infty), \quad \text{for all } f \in K_0.$$

Also let λ be such that

$$\int_{\{|f|>\lambda\}} |f|\, d\mu < \varepsilon/(8\|g\|_\infty), \quad \text{for all } f \in K_0.$$

Since $\chi_C g' \in L^1_{E'}(X_0, \Sigma_0, \mu)$, we have $\lim_n T^{h_n}(\chi_C g') = \chi_C g'$, strongly in $L^1_{E'}$; and we have also

$$\lim_n \|\chi_C - \chi_{C-h_n}\|_1 = \lim_n \|\chi_C - T^{h_n}\chi_C\|_1 = 0.$$

Let n_ε be such that for $n \geq n_\varepsilon$ we have

$$\|T^{h_n}(\chi_C g') - \chi_C g'\|_1 < \varepsilon/(4\lambda)$$

and

$$\|\chi_C - \chi_{C-h}\|_1 < \varepsilon/(4\lambda\|g\|_\infty)$$

Then, for any $f \in K_0$ and any $n \geqslant n_\varepsilon$ we deduce from step (F),

$$\left|\int \langle T^{h_n} f, g\rangle \, d\mu\right| = \int \langle T^h f, g'\rangle \, d\mu | < \varepsilon,$$

that is $\lim_n T^{h_n} f = f$, in L^1_E for the σ'-topology, uniformly for $f \in K_0$. Since the sequence (h_n) was arbitrary, it follows that

$$\lim_{h \in V_n,\, n \to \infty} T^h f = f, \quad \text{in } L^1_E$$

for the σ'-topology, uniformly for $f \in K$ and so, condition (b_2) is proved. Condition (b_1) then follows from Proposition 12 in **[4]**; and this completes the proof of the theorem.

REMARK. The σ'-topology cannot be replaced by the weak topology. There are examples (to be published in a joint paper with Jürgen Batt) of relatively weakly compact sets $K \subset L^1_E$ over the circle group, such that the limits in (b_1) and (b_2) for the weak topology are false.

3. Uniform σ-additivity in the Pettis space $\mathfrak{L}^1_E$. We denote by $\mathfrak{L}^1_E$ the Pettis space of functions $f\colon G \to E$ which are strongly μ-measurable and Pettis integrable, endowed with the Pettis norm

$$(f)_1 = \sup\left\{\int |\langle f, x'\rangle| \, d\mu;\ x' \in E'_1\right\}$$

where E'_1 is the unit ball of E'. A set $F \subset E'_1$ is norming for a set $K \subset \mathfrak{L}^1_E$, if $|f(t)| = \sup\{|\langle f(t), x'\rangle| ;\ x' \subset F\}$, μ-a.e. for every $f \in K$.

THEOREM 2. *Let $K \subset \mathfrak{L}^1_E$ be a set.*

I. *The set $K\mu$ is uniformly σ-additive, if and only if the following conditions are satisfied*:

(a) *K is bounded in $\mathfrak{L}^1_E$*;

(b) *For every countable subset $K_0 \subset K$ there is a decreasing sequence (V_n) of neighborhoods of* 0 *and a countable subset $E'_0 \subset E'_1$, norming for K_0, such that either*

(b_1) $\lim_n \langle u_{V_n} * f, x'\rangle = \langle f, x'\rangle$, *weakly in L^1, uniformly for $f \in K_0$ and $x' \in E'_0$*;

or

(b_2) $\lim_{h \in V_n, n \to \infty} \langle T^h f, x'\rangle = \langle f, x'\rangle$, *weakly in L^1, uniformly for $f \in K_0$ and $x' \in E'_0$*;

(c) $\lim_C f\chi_C = f$, *strongly in $\mathfrak{L}^1_E$, uniformly for $f \in K$, the limit being taken along the increasing net of all compact subsets of G.*

II. *If $K\mu$ is uniformly σ-additive, then*:

(b'_1) $\lim_V \langle u_V * f, x'\rangle = \langle f, x'\rangle$ *and*

(b'_2) $\lim_{h \to 0} \langle T^h f, x'\rangle = \langle f, x'\rangle$

weakly in L^1, uniformly for $f \in K$ and $x' \in E'_1$.

PROOF. Assume first conditions (a), (b) and (c) satisfied. Let $K_0 \subset K$ be a countable set, and E'_0 the set corresponding to K_0 by condition (b) above.

Then the set $\langle K_0, E'_0\rangle = \{\langle f, x'\rangle;\ f \in K_0,\ x' \in E'_0\}$ is countable and satisfies conditions (a), (b) and (c) of Theorem 1, in the space L^1. It follows that $\langle K_0, E'_0\rangle\mu$ is uniformly σ-additive; and then $K_0\mu$ is also uniformly σ-additive; therefore $K\mu$ is uniformly σ-additive.

Conversely, assume $K\mu$ is uniformly σ-additive; let $K_0 \subset K$ be countable, and let $E'_0 \subset E'_1$ be a countable set, norming for K_0. The set $\langle K_0, E'_0 \rangle$ is countable and uniformly σ-additive. From Theorem 1 we deduce:

(a) The set $\langle K_0, E'_0 \rangle$ is bounded in L^1; hence K_0 is bounded in $\mathfrak{L}^1_E$; therefore K is bounded in $\mathfrak{L}^1_E$;

(b) There exists a decreasing sequence (V_n) of neighborhoods of 0, satisfying conditions (b_1) and (b_2) of this theorem;

(c) $\lim_C \langle \chi_C f, x' \rangle = \langle f, x' \rangle$, strongly in L^1 uniformly for $f \in K$ and $x' \in E'_0$, which is equivalent to condition (c) of this theorem.

Finally, to obtain (b'_1) and (b'_2) we apply the second part of Theorem 1 to the set $\langle K, E'_1 \rangle$.

Note. We take this opportunity to mention that Theorems 2(iii) and 4(iii) in **[3]** are valid without the condition $\sup\{|f(t)|; f \in K\} < \infty$, μ-a.e. The proof will be given in a forthcoming paper, for a more general situation.

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