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SOME PROPERTIES OF BOREL SUBGROUPS OF REAL NUMBERS

BARTHÉLEMY LE GAC

ABSTRACT. As a consequence of Souslin's theorem, we obtain the following: if G and H both are analytic subgroups of $\mathbf{R}$ such that $G + H = \mathbf{R}$ and $G \cap H = \{0\}$, then either $G = \mathbf{R}$ or $G = \{0\}$. Next we obtain some measure and topological properties for uncountable proper Borel subgroups of reals. Finally, we prove that if E is a vector subspace of $\mathbf{R}$ over the rationals which admits an uncountable Borel basis, then there exists no Polish topology on E such that E is a topological group with the given Borel structure generated by the open sets.

1. Introduction. In [**12**], J. von Neumann constructed a set A of real numbers which is algebraically independent (over the field of rational numbers $\mathbf{Q}$) and has the power of continuum. The elements of A are the reals $f(x)$ given by

$$f(x) = \sum_{n=0}^{\infty} 2^{2^{\operatorname{Int}(nx)}}/2^{2^{n^2}}, \qquad x > 0.$$

This function f is strictly increasing, so one-one and Borel; hence A is a Borel set by Lusin's theorem [**8**, p. 176]. If P is a proper nonvoid perfect compact subset of A, the vector subspace G of $\mathbf{R}$ over the rationals generated by P gives us an example of an uncountable proper Borel subgroup (in fact a σ-compact one) of the reals. More generally, J. Mycielski proved in [**11**, Theorem 2] that every nonvoid perfect set of reals contains a nonvoid algebraically independent perfect subset; this is the key theoretic tool for the construction of such subgroups of $\mathbf{R}$. Finally, let us recall that any proper analytic subgroup of $\mathbf{R}$, both is a meager and a null set. (But note that a null subgroup of the reals is not always a meager set, see [**4**].)

2. A consequence of Souslin's theorem.

THEOREM 1. *If G and H both are analytic subgroups of $\mathbf{R}$ such that $G + H = \mathbf{R}$ and $G \cap H = \{0\}$, then either $G = \mathbf{R}$ or $G = \{0\}$.*

PROOF. We equip both G and H with their relativized Borel structure. The map f of $G \times H$ onto $\mathbf{R}$ defined by $f(x, y) = x + y$ is one-one and Borel; hence it is a Borel isomorphism, in virtue of Souslin's theorem [**3**, p. 135]. Let g denote the inverse map of f; we have $g(t) = (x(t), y(t))$, for every real t. Hence both x and y are Borel

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group homomorphisms; so they are continuous, by virtue of a theorem by Banach which asserts that every Borel homomorphism of a Polish group E to a separable metric group F is continuous [**1**, Theorem 4, p. 23]. Then both G and H are closed in $\mathbf{R}$. But one knows that a closed proper subgroup of the reals takes the form of an $a\mathbf{Z}$, for some real a. Now we conclude that either $G = \mathbf{R}$ or $G = \{0\}$.

REMARK. This theorem fails if H is supposed only to be a null subgroup; see [**5**, p. 192], in which the authors exhibit such an H which is moreover a Lusin set, if *continuum hypothesis* holds.

COROLLARY. *Let K be an analytic proper subfield of the reals. There is no analytic basis A for $\mathbf{R}$ as a K vector space.*

PROOF. Suppose that such an A exists; hence for any a in A, both the subspaces G and H generated respectively by $\{a\}$ and $A \setminus \{a\}$ are analytic subgroups of $\mathbf{R}$. But this is impossible, by virtue of the preceding theorem.

In the case $K = \mathbf{Q}$, this corollary is in fact the classical result by Sierpiński: there is no Hamel basis in $\mathbf{R}$ which is an analytic set [**13**, p. 110].

Let us remark that if K is a nonanalytic subfield of the reals, there may exist analytic bases for $\mathbf{R}$ as a K vector space; see [**3**], for an example with such a countable (infinite) basis.

3. A measure theoretic result. Throughout this section, we denote by G an uncountable proper Borel subgroup of the reals, equipped with its given Borel structure (the relativized one).

THEOREM 2. *There exists no nonzero Borel measure μ on G which is quasi-invariant (for the action of G on G).*

PROOF. Let us suppose that such a measure μ exists on G. By the Weil-Mackey theorem [**9**, p. 146], there exists a unique locally compact second countable topology on G such that G is a topological group with the given Borel structure generated by the open sets. This topology is not discrete, since G is uncountable. As a locally compact abelian group, G is (topologically) isomorphic to a direct product $\mathbf{R}^n \times H$, in which H contains a compact open subgroup [**6**, Theorem 3.2, p. 69]. We have necessarily $n = 0$, otherwise we can inject the connected group $\mathbf{R}^n \times \{0\}$ into G. Hence G is isomorphic to H, and so it contains a compact open subgroup K; but this K will also be a compact subgroup of $\mathbf{R}$, which implies $K = \{0\}$. Now the zero subgroup is open in G, which is impossible since the topology on G is not the discrete one.

COROLLARY. *There exists no locally compact topology on G such that G is a topological group with the given Borel structure generated by the open sets.*

PROOF. If such a topology exists on G, it contains a compact open subgroup K, by the same argument as above. This K cannot be uncountable, in virtue of the precedent theorem; hence K is finite, since it is both compact and countable; so K is the zero subgroup in the reals. Now the group G both is uncountable and discrete, which is impossible since the Borel sets in G are not the whole subsets of G.

4. In this last section, we are concerned only with vector subspaces of $\mathbf{R}$ over the rationals which admit Borel bases; such vector subspaces are Borel sets in $\mathbf{R}$; this is a result by R. D. Mauldin [**10**, p. 263]; using the same argument, we also obtain the following.

PROPOSITION. *Let E and E' both be vector subspaces of* $\mathbf{R}$, *over the rationals, both of which admit respective Borel bases B and B' such that* $\operatorname{card}(B) = \operatorname{card}(B')$. *Then E and E' are isomorphic, as Borel (standard) groups.*

PROOF. We may restrict our attention to the case in which both B and B' have the power of continuum. For every integer $n > 0$, let B_n be the Borel subset of the n-copies product B^n, the elements of which are the n-tuples of all different basis vectors. Let us define a map f_n of $B_n \times \mathbf{Q}^n$ to $\mathbf{R}$ by

$$f_n(b_1,\ldots,b_n;\, r_1,\ldots,r_n) = \sum r_i b_i.$$

Hence the set $E_n = f_n(B_n \times \mathbf{Q}^n)$ is Borel (Mauldin, [**10**]), and E is equal to the increasing union of the sequence of Borel sets E_n. Using a result by R. R. Kallman [**7**, Proposition 7.2, p. 240], we can obtain a Borel map s_n of E_n to $B_n \times \mathbf{Q}^n$ such that if

$$s_n(t) = (b_1(t),\ldots,b_n(t);\, r_1(t),\ldots,r_n(t)),$$

then we have $\Sigma r_i(t) b_i(t) = t$, for all t in E_n.

Let us consider now both the bases B and B'; there exists a Borel isomorphism g of B onto B' [**8**, p. 114]; we may naturally associate to it a g_n of $B_n \times \mathbf{Q}^n$ onto $B_{n'} \times \mathbf{Q}^n$, given by

$$g_n(b_1,\ldots,b_n;\, r_1,\ldots,r_n) = (g(b_1),\ldots,g(b_n);\, r_1,\ldots,r_n).$$

(We define B_n', f_n' and E_n' related to E' in the same way as the corresponding ones related to E.) Now we see that the composed map $h_n = f_n' \circ g_n \circ s_n$ is one-one and Borel of E_n onto E_n', and so it is a Borel isomorphism, by Souslin's thereom. Let t in E_n be given by $t = \Sigma r_i b_i$; thus $h_n(t) = \Sigma r_i g(b_i)$; so if we define a map h of E to E' by $h(t) = h_n(t)$, whenever t is in E_n, we will obtain the desired Borel groups isomorphism.

THEOREM 3. *Let E be a vector subspace of* $\mathbf{R}$ *over the rationals which admits an uncountable Borel basis B. Then there exists no Polish topology on E such that G is a topological group with the given Borel structure generated by the open sets.*

PROOF. (The following proof is Mauldin's, the original one by the author being much more complicated.) Let us suppose that such a topology exists on E. Write B as a strictly increasing union of a sequence of Borel sets B_n, and let E_n be the corresponding vector subspace generated by B_n. We may assume that E_1 is dense in E (equipped with its Polish topology). Each E_n is an analytic subgroup of E (in fact a Borel one), and therefore has the Baire property. One of the E_n's must be of the second category; hence this E_n is closed and open in E, by [**1**, Theorem 1, p. 21]. It follows that this E_n is all of E, since it is also dense in E.

Question. Is there an uncountable proper Borel subgroup G of $\mathbf{R}$ and a Polish topology on G such that G is a topological group with the given Borel structure generated by the open sets? •

(If such a topology exists on a certain G, it is necessarily unique, by [**1**, Theorem 4, p. 23].)

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U. E. R. DE MATHÉMATIQUES, UNIVERSITÉ DE PROVENCE, 3, PLACE VICTOR-HUGO, 13331 MARSEILLE CEDEX 3, FRANCE