PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 4, April 1983

FOURIER COEFFICIENTS OF CONTINUOUS FUNCTIONS ON COMPACT GROUPS

BARBARA HEIMAN

ABSTRACT. Let G be an infinite compact group with dual object Σ. Letting $\mathcal{H}_\sigma$ be the representation space for $\sigma \in \Sigma$, $\mathfrak{S}^2(\Sigma)$ is the set $\{A = (A^\sigma) \in \prod \mathfrak{B}(\mathcal{H}_\sigma)\colon \|A\|_2^2 = \sum_\sigma d_\sigma \operatorname{Tr}(A^\sigma A^{\sigma^*}) < \infty\}$. For $A \in \mathfrak{S}^2(\Sigma)$, we show that there is a function f in $C(G)$ such that $\|f\|_\infty \leqslant C\|A\|_2$ and $\operatorname{Tr}(\hat{f}(\sigma)\hat{f}(\sigma)^*) \geqslant \operatorname{Tr}(A^\sigma A^{\sigma^*})$ for every $\sigma \in \Sigma$.

In a 1977 paper [3], K. de Leeuw, Y. Katznelson and J.-P. Kahane proved that every square summable sequence is dominated by the sequence of Fourier coefficients of a continuous function on the circle group, **T**. As the authors mentioned, this result is true, with the same proof, for any compact abelian group in the role of **T** and its dual group in place of the integers, **Z**. This paper answers the same question for a compact nonabelian group. Using appropriate tools, our proof parallels that of [3]. All notation and terminology used here without explicit definition is as in [2].

Let G be an infinite compact group with dual object Σ. For each $\sigma \in \Sigma$, let U^σ be a representation in σ and let $\mathcal{H}_\sigma$, its representation space, have dimension d_σ. If $\mathfrak{B}(\mathcal{H}_\sigma)$ is the space of operators on $\mathcal{H}_\sigma$, define $\|\ \|_2$ on $\mathfrak{B}(\mathcal{H}_\sigma)$ by $\|A^\sigma\|_2 = \operatorname{Tr}(A^\sigma A^{\sigma^*})^{1/2}$. Let $\mathfrak{S}(\Sigma) = \prod_{\sigma\in\Sigma} \mathfrak{B}(\mathcal{H}_\sigma)$ and let $\mathfrak{S}^2(\Sigma)$ be the set of $A = (A^\sigma) \in \mathfrak{S}(\Sigma)$ satisfying

$$\|A\|_2 = \left(\sum_\sigma d_\sigma \|A^\sigma\|_2^2\right)^{1/2} < \infty.$$

Finally, Γ will designate the compact group $\prod_{\sigma\in\Sigma} \mathfrak{U}(d_\sigma)$, where $\mathfrak{U}(d_\sigma)$ is the group of all unitary operators on $\mathcal{H}_\sigma$.

We make use of the following results.

(1) Let $f(V) = \sum_\sigma d_\sigma \operatorname{Tr}(B^\sigma V^\sigma)$ $(V \in \Gamma)$ be a finite sum. Then

$$\int_\Gamma |\exp f(V)|\,dV \leqslant \exp\left(\|B\|_2^2\right).$$

This statement and its proof are similar to [4, Lemma 2].

(2) Suppose $A \in \mathfrak{S}^2(\Sigma)$. Then, for almost all $V \in \Gamma$,

$$\sum_\sigma d_\sigma \operatorname{Tr}(A^\sigma V^\sigma U^\sigma(x))$$

converges for almost every $x \in G$ [4, Lemma 8].

Received by the editors May 24, 1982.

1980 *Mathematics Subject Classification*. Primary 43A30, 43A77, 60B15.

LEMMA 1. *Suppose $A \in \mathcal{E}^2(\Sigma)$. For every $V \in \Gamma$, let f_V be the function in $L^2(G)$ whose Fourier series is*

$$\sum_\sigma d_\sigma \operatorname{Tr}(A^\sigma V^\sigma U^\sigma(x)).$$

Let $\lambda > 0$. Then the set

$$E = \left\{ V \in \Gamma \colon \int_G \exp \lambda |f_V(x)| \, dx \leqslant 2 \exp\left(4\lambda^2 \|A\|_2^2\right) \right\}$$

has positive Γ-measure.

PROOF. Let $x \in G$ be fixed but arbitrary. Let $S_n f_V$ be the nth partial sum of f_V and S_n, that of $\|A\|_2^2$. (These partial sums are relative to some arbitrary and fixed enumeration of the set $\{\sigma \in \Sigma \colon A^\sigma \neq 0\}$.)

Using the inequality $\exp|z| \leqslant \sum_{k=1}^4 \frac{1}{2} |\exp 2i^k z|$, the invariance of the Haar integral on Γ and (1),

$$\int_\Gamma \exp \lambda |S_n f_V(x)| \, dV \leqslant 2 \exp(4\lambda^2 S_n).$$

As $x \in G$ was arbitrary,

$$\int_G \int_\Gamma \exp \lambda |S_n f_V(x)| \, dV \, dx \leqslant 2 \exp(4\lambda^2 S_n).$$

By Fubini's theorem, the order of integration can be reversed. Applying (2) and Fatou's lemma,

$$\int_\Gamma \int_G \exp \lambda |f_V(x)| \, dx \, dV \leqslant 2 \exp\left(4\lambda^2 \|A\|_2^2\right).$$

Consequently, E has positive Γ-measure.

LEMMA 2. *Suppose $B \in \mathcal{E}^2(\Sigma)$. Let $\varepsilon, \eta > 0$ be given and suppose that $\|B\|_2 \leqslant \varepsilon$. Then there is a choice of $V \in \Gamma$ such that the function $h \in L^2(G)$, whose Fourier series is $\sum_\sigma d_\sigma \operatorname{Tr}(B^\sigma V^\sigma U^\sigma(x))$, satisfies*

$$\left\| (|h| - \eta)^+ \right\|_2 \leqslant 16\sqrt{2}\, e^{-1} \varepsilon^2 \eta^{-1} \exp\left(\frac{-\eta^2}{32\varepsilon^2} \right).$$

PROOF. Let $\lambda = \eta / 8\varepsilon^2$. Invoke Lemma 1 to obtain $V \in \Gamma$ so that

$$\int_G \exp(\lambda |h(x)|) \, dx \leqslant 2 \exp\left(4\lambda^2 \|B\|_2^2\right) \leqslant 2 \exp\left(\frac{\eta^2}{16\varepsilon^2} \right).$$

Since

$$\sup_{t \geqslant \eta} (t - \eta)^2 \exp(-\lambda t) = 256 e^{-2} \varepsilon^4 \eta^{-2} \exp\left(\frac{-\eta^2}{8\varepsilon^2} \right),$$

$$\begin{aligned} \left\| (|h| - \eta)^+ \right\|_2^2 &\leqslant 256 e^{-2} \varepsilon^4 \eta^{-2} \exp\left(\frac{-\eta^2}{8\varepsilon^2} \right) \int_G \exp(\lambda |h(x)|) \, dx \\ &\leqslant 512 e^{-2} \varepsilon^4 \eta^{-2} \exp\left(\frac{-\eta^2}{16\varepsilon^2} \right). \end{aligned}$$

LEMMA 3. *Suppose $A \in \mathcal{E}^2(\Sigma)$. There exists a function $f \in L^\infty(G)$ with $\|f\|_\infty \leq 36\|A\|_2$, and $\|\hat{f}(\sigma)\|_2 \geq \|A^\sigma\|_2$ for every $\sigma \in \Sigma$.*

PROOF. Assume that $\|A\|_2 = 1$. Define sequences $(\delta_j)_{j=1}^\infty$, $(\eta_j)_{j=1}^\infty$, and $(\varepsilon_j)_{j=0}^\infty$ by

$$\delta_j = 3^{-j}; \quad \eta_j = 36\delta_j; \quad \varepsilon_0 = 1$$

and

$$\varepsilon_{j+1} = 32\sqrt{2}\,(1-\delta_1)e^{-1}\varepsilon_j^2\eta_{j+1}^{-1}\delta_{j+1}^{-1}\exp\left(\frac{-n_{j+1}^2}{32\varepsilon_j^2}\right) \quad \text{for } j \geq 0.$$

One checks by induction that $\varepsilon_j \leq 6^{-j}$ for $j \geq 0$. Let $s_0 = 0$ and $s_k = \sum_{j=1}^k \delta_j$.

We next define sequences of functions $(f_j)_{j=0}^\infty$, $(g_j)_{j=0}^\infty$ and $(h_j)_{j=0}^\infty$ with $f_j = g_j + h_j$ which satisfy

(a) $\|g_j\|_\infty \leq 36 s_j$;

(b) $\|h_j\|_2 \leq \varepsilon_j$;

(c) $\left\|\left(|h_j| - \eta_{j+1}\right)^+\right\|_2 \leq \rho_j = 16\sqrt{2}\,e^{-1}\varepsilon_j^2\eta_{j+1}^{-1}\exp\left(\frac{-\eta_{j+1}^2}{32\varepsilon_j^2}\right)$;

(d) $\|\hat{f}_j(\sigma)\|_2 \geq (1 - s_j)\|A^\sigma\|_2$ for $\sigma \in \Sigma$.

Let $g_0 = 0$ and choose h_0 to satisfy the conclusion of Lemma 2 with $B = A$, $\varepsilon = \varepsilon_0$ and $\eta = \eta_1$. Then (a)–(d) are true for $j = 0$.

Suppose that $k \geq 1$ and that g_{k-1} and h_{k-1} have been selected to satisfy (a)–(d) when $j = k - 1$. Define

$$g_k(x) = \begin{cases} 36 s_k \operatorname{sgn} f_{k-1}(x) & \text{if } |f_{k-1}(x)| > 36 s_k, \\ f_{k-1}(x) & \text{if } |f_{k-1}(x)| \leq 36 s_k. \end{cases}$$

Then (a) holds for $j = k$. Note that if $|f_{k-1}(x)| > 36 s_k$, then

$$\begin{aligned} |f_{k-1}(x) - g_k(x)| &= |f_{k-1}(x)| - 36 s_k \\ &\leq |g_{k-1}(x)| - 36 s_{k-1} + |h_{k-1}(x)| - \eta_k \\ &\leq \left(|h_{k-1}(x)| - \eta_k\right)^+. \end{aligned}$$

Thus $\|f_{k-1} - g_k\|_2 \leq \rho_{k-1}$. Let

$$\Phi = \left\{\sigma \in \Sigma \colon \|\hat{g}_k(\sigma)\|_2 < (1 - s_k)\|A^\sigma\|_2\right\}.$$

For $\sigma \in \Phi$, we have $\|\hat{f}_{k-1}(\sigma) - \hat{g}_k(\sigma)\|_2 \geq \delta_k \|A^\sigma\|_2$. Define $B \in \mathcal{E}^2(\Sigma)$ by

$$B^\sigma = \begin{cases} 2(1 - s_k)A^\sigma & \text{for } \sigma \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\|B\|_2^2 = \sum_{\sigma \in \Phi} d_\sigma 4(1 - s_k)^2 \|A^\sigma\|_2^2$$

$$\leqslant 4(1 - s_k)^2 \delta_k^{-2} \|f_{k-1} - g_k\|_2^2$$

$$\leqslant 4(1 - \delta_1)^2 \delta_k^{-2} \rho_{k-1}^2 = \varepsilon_k^2.$$

Thus a function h_k can be chosen via Lemma 2 applied to B, $\varepsilon = \varepsilon_k$ and $\eta = \eta_{k+1}$ which satisfies both (b) and (c). Finally, if $\sigma \notin \Phi$,

$$\|\hat{f}_k(\sigma)\|_2 = \|\hat{g}_k(\sigma)\|_2 \geqslant (1 - s_k)\|A^\sigma\|_2,$$

while if $\sigma \in \Phi$,

$$\|\hat{f}_k(\sigma)\|_2 \geqslant \|B^\sigma\|_2 - \|\hat{g}_k(\sigma)\|_2 \geqslant (1 - s_k)\|A^\sigma\|_2.$$

Therefore (d) holds for $j = k$. This completes the definition of these sequences of functions.

For $j \geqslant 1$,

$$\|f_{j-1} - f_j\|_2 \leqslant \|f_{j-1} - g_j\|_2 + \|h_j\|_2 < 2(6^{-j}),$$

so there is a function $f \in L^2(G)$ such that $\lim_{j \to \infty} \|2f_j - f\|_2 = 0$. Since $\|2g_j - f\|_2 \leqslant \|2f_j - f\|_2 + 2\|h_j\|_2$,

$$\lim_{j \to \infty} \|2g_j - f\|_2 = 0.$$

Thus a subsequence of $(2g_j)_{j=0}^{\infty}$ converges to f pointwise almost everywhere on G. Hence

$$\|f\|_\infty \leqslant 2 \overline{\lim_{j \to \infty}} \|g_j\|_\infty \leqslant 36$$

and, for each $\sigma \in \Sigma$,

$$\|\hat{f}(\sigma)\|_2 \geqslant 2 \lim_{j \to \infty} (1 - s_j)\|A^\sigma\|_2 = \|A^\sigma\|_2.$$

This verifies the lemma.

We are now able to state and prove the main result, which replaces $L^\infty(G)$ in Lemma 3 by $C(G)$.

THEOREM. *Suppose $A \in \mathfrak{S}^2(\Sigma)$. There exists a function $f \in C(G)$ with $\|f\|_\infty \leqslant 37\|A\|_2$ and $\|\hat{f}(\sigma)\|_2 \geqslant \|A^\sigma\|_2$ for every $\sigma \in \Sigma$.*

PROOF. Let $\delta = \|A\|_2/36$. Assume $\delta > 0$. Let $h \in L^2(G)$ have Fourier series

$$\sum_\sigma d_\sigma \operatorname{Tr}(A^\sigma U^\sigma(X)).$$

By a factorization theorem, there exist functions $g \in L^2(G)$ and $k \in L^1(G)$ such that

$h = k * g$;

k is nonnegative and central in $L^1(G)$;

$\|k\|_1 = 1$;

$\|h - g\|_2 < \delta$.

(See [**2**, (32.31)], replacing $C(G)$ by $L^2(G)$.)

Invoke Lemma 3 to obtain a function $f_\infty \in L^\infty(G)$ which satisfies $\|f_\infty\|_\infty \leq 36\|g\|_2$ and $\|\hat{f}_\infty(\sigma)\|_2 \geq \|\hat{g}(\sigma)\|_2$ for every $\sigma \in \Sigma$. Let $f = k * f_\infty \in L^1(G) * L^\infty(G) = C(G)$. Then

$$\|f\|_\infty \leq \|k\|_1 \|f_\infty\|_\infty \leq 36\|g\|_2 \leq 36(\|h\|_2 + \delta) = 37\|A\|_2.$$

Since k is central in $L^1(G)$, $\hat{k}(\sigma)$ is seen to be a scalar multiple of the identity in $\mathfrak{B}(\mathcal{H}_\sigma)$. Write $\hat{k}(\sigma) = c_\sigma I_\sigma$. Then

$$\|\hat{f}(\sigma)\|_2 = |c_\sigma| \, \|\hat{f}_\infty(\sigma)\|_2 \geq |c_\sigma| \, \|\hat{g}(\sigma)\|_2 = \|A^\sigma\|_2.$$

A corollary of our theorem is a generalization of Carleman's theorem for the circle [**1**]. (See also [**2**, 37.22(k)] for another proof of this corollary.) For the statement of this corollary, some additional notation is necessary. Let $A \in \mathfrak{B}(\mathcal{H}_\sigma)$ and let $|A|$ be the (unique) positive-definite square root of AA^*. Let $(x_1, \ldots, x_{d_\sigma})$ be the eigenvalues of $|A|$. Define the *von Neumann* norms on $\mathfrak{B}(\mathcal{H}_\sigma)$ by

$$\|A\|_{\phi_p} = \Big(\sum_j (x_j)^p\Big)^{1/p} \qquad (1 \leq p < \infty).$$

(Note that $\|\ \|_{\phi_2}$ is the same as $\|\ \|_2$.) For $A = (A^\sigma) \in \mathfrak{S}(\Sigma)$, define

$$\|A\|_p = \Big(\sum_\sigma d_\sigma \|A^\sigma\|_{\phi_p}^p\Big)^{1/p} \qquad (1 \leq p < \infty).$$

Let $\mathfrak{S}^p(\Sigma) = \{A \in \mathfrak{S}(\Sigma) \colon \|A\|_p < \infty\}$.

COROLLARY. *There is a continuous function f defined on G for which $\hat{f} \notin \mathfrak{S}^p(\Sigma)$ for $1 \leq p < 2$.*

PROOF. Let $\{\sigma_2, \sigma_3, \sigma_4, \ldots\}$ be a countably infinite subset of Σ, with no repetitions. Let A^{σ_n} be the $d_{\sigma_n} \times d_{\sigma_n}$-matrix whose $(1, 1)$-entry is $1/\sqrt{nd_{\sigma_n}} \log n$ and whose other entries are zero. Let $A^\sigma = 0$ for all other $\sigma \in \Sigma$. Then $A \in \mathfrak{S}^2(\Sigma)$. By the preceding theorem, there exists a function $f \in C(G)$ with

$$\|\hat{f}(\sigma)\|_2 \geq \|A^\sigma\|_2 \quad \text{for every } \sigma \in \Sigma.$$

For every $\sigma \in \Sigma$,

$$\|\hat{f}(\sigma)\|_{\phi_p} \geq \|\hat{f}(\sigma)\|_2 \geq \|A^\sigma\|_2 = \|A^\sigma\|_{\phi_p}.$$

Since

$$d_{\sigma_n}\|A^{\sigma_n}\|_{\phi_p}^p = d_{\sigma_n}\left(\frac{1}{\sqrt{nd_{\sigma_n}} \log n}\right)^p \geq \frac{1}{n \log n}$$

for all sufficiently large n, $\hat{f} \notin \mathfrak{S}^p(\Sigma)$.

ACKNOWLEDGEMENTS. An earlier version of this paper was written while the author was a graduate student at Kansas State University. I am grateful to the mathematics department there for their support and encouragement. Special thanks are due Karl Stromberg for his guidance. I would also like to thank Edwin Hewitt for his valuable suggestions.

References

1. T. Carleman, *Über die Fourierkoeffizienten einer stetigen Funktion*, Acta Math. **41** (1918), 377–384.

2. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. II, Grundlehren. Math. Wiss. No. 152, Springer-Verlag, Berlin, Heidelberg and New York, 1970.

3. K. de Leeuw, Y. Katznelson and J.-P. Kahane, *Sur les coefficients de Fourier des fonctions continues*, C. R. Acad. Sci. Paris Sér. A. **285** (1977), 1001–1003.

4. D. Rider, *Random fourier series*, Symposia Mathematica No. 22, Academic Press, New York, 1977, pp. 93–106.

Department of Mathematical Sciences, The College of Wooster, Wooster, Ohio 44691