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ON GROUP D.G. NEAR-RINGS

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ABSTRACT. Meldrum has generalized the idea of a group ring and has defined a group d.g. near-ring for a faithful d.g. near-ring (R, S) on a multiplicative group G. In this paper we generalize this idea even further and define a group d.g. near-ring for an arbitrary d.g. near-ring. We also prove some results about the additive group of this d.g. near-ring similar to those proved by Meldrum for a group d.g. near-ring of a faithful d.g. near-ring.

1. Preliminaries. A set R, together with two binary operations $+$ and $\cdot$, is called a (left) near-ring if:

(i) $(R, +)$ is a group (not necessarily abelian);

(ii) $(R, \cdot)$ is a semigroup;

(iii) $x(y+z) = xy + xz$ for all $x, y, z \in R$.

An element $d \in R$ is called distributive if $(x+y)d = xd + yd$ for all $x, y \in R$. The subset D of distributive elements forms a subsemigroup of $(R, \cdot)$.

R is called a distributively generated (d.g.) near-ring if $(R, +)$ is generated by a distributive semigroup S which is not necessarily the whole set of distributive elements of R. A d.g. near-ring is denoted by (R, S). We call θ: $(R, S) \to (T, U)$ a d.g. near-ring homomorphism if θ: $(R, +) \to (T, +)$ is a group homomorphism and θ: $(R, \cdot) \to (T, \cdot)$ is a semigroup homomorphism such that $S\theta \subseteq U$. A semigroup homomorphism θ: $S \to U$ is a d.g. near-ring homomorphism from $(R, S) \to (T, U)$ if and only if it is a group homomorphism from $(R, +)$ to $(T, +)$. From now on we will use the term homomorphism for a d.g. near-ring homomorphism unless otherwise stated.

If, for a group G, θ: $(R, S) \to (E(G), \operatorname{End} G)$ is a homomorphism, then θ is called a d.g. representation of (R, S) on G. Here $E(G)$ is the d.g. near-ring of mappings from G to itself generated by $\operatorname{End} G$, the set of all endomorphisms of G. A d.g. near-ring is called faithful if it has a faithful d.g. representation, i.e., $\operatorname{Ker} \theta = \{0\}$. Not all d.g. near-rings are faithful **[3]**. (R, S) is faithful if and only if an identity 1 can be adjoined to R such that the elements of S remain distributive in the bigger d.g. near-ring **[5]**. However, with every d.g. near-ring we can associate two faithful d.g. near-rings (Meldrum **[3]**, Mahmood **[2]**). The upper faithful d.g. near-ring for (R, S) is a faithful d.g. near-ring $(\bar{R}, S)$ together with an epimorphism θ: $(\bar{R}, S) \twoheadrightarrow (R, S)$ such that (i) $\theta|_S = 1_S$, (ii) if ϕ: $(T, U) \to (R, S)$ is a homomorphism, where (T, U) is faithful, then there exists a unique homomorphism ψ: $(T, U) \to (\bar{R}, S)$ such that $\psi\theta = \phi$. The lower faithful d.g. near-ring for (R, S) is a

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faithful d.g. near-ring $(\underline{R}, S)$ together with an epimorphism $\underline{\theta}$: $(R, S) \twoheadrightarrow (\underline{R}, \underline{S})$ such that (i) $S\underline{\theta} = \underline{S}$, (ii) if ϕ: $(R, S) \to (T, U)$ is a homomorphism, where (T, U) is faithful, then there exists a unique homomorphism ψ: $(\underline{R}, \underline{S}) \to (T, U)$ such that $\underline{\theta}\psi = \phi$.

We will be using the following two results from [3]. For each (R, S) group H, $\operatorname{Ker}\underline{\theta} \subseteq \operatorname{Ann}_R H$. Let $X \subseteq R$, where (R, S) is a d.g. near-ring. Then $\operatorname{Id}\langle X\rangle$, the ideal generated by X, is the normal subgroup of $(R, +)$ generated by $RXS = \{rxs, rx, xs, x;\ r \in R, x \in X, s \in S\}$. We now present basic facts about d.g. near-rings taken from [4]. Let (R, S) be a faithful d.g. near-ring and let G be a multiplicative group. Let X be any set, $Y = X \times G = \{(x, g);\ x \in X, g \in G\}$, and $F = \operatorname{Fr}(Y, R, S)$ be the free (R, S) group on the set Y [3]. Then by means of right regular representations, G can be defined as a group of (R, S) automorphisms of F. So the semigroup $SG = \{sg;\ s \in S, g \in G\}$ of endomorphisms of F generates a d.g. near-ring $(R(G), SG)$ in $E(F)$. This d.g. near-ring is defined to be the group d.g. near-ring of (R, S) on G. Also $rg = gr$ in $R(G)$ for all $r \in R$, $g \in G$. F is the free $(R(G), SG)$ group on the set X and $(R(G), +)$ considered as an (R, S) group is an orthogonal sum of its (R, S) subgroups $\{Rg;\ g \in G\}$.

The idea of an orthogonal sum comes from Fröhlich [1]. If $\{H_\lambda;\ \lambda \in \Lambda\}$ is a family of (R, S) groups, then H is an orthogonal sum of $\{H_\lambda;\ \lambda \in \Lambda\}$ if it is an (R, S) group, and (R, S) homomorphisms α_λ, β_λ exist for all $\lambda \in \Lambda$ such that α_λ: $H_\lambda \to H$, β_λ: $H \to H_\lambda$ and $\alpha_\lambda\beta_\mu$ is the identity map on H_λ if $\lambda = \mu$, and is the zero map otherwise. Note that this forces α_λ to be a monomorphism and β_λ an epimorphism for all $\lambda \in \Lambda$. We add the condition that $H = Gp\langle H_\lambda\alpha_\lambda;\ \lambda \in \Lambda\rangle$. Fröhlich calls this a covered orthogonal sum. This is equivalent to saying that there exist homomorphisms $\theta, \phi {*}_{\lambda\in\Lambda} H_\lambda \xrightarrow{\theta} H \xrightarrow{\phi} \oplus_{\lambda\in\Lambda} H_\lambda$, where $*$ indicates the free (R, S) product, $\oplus$ indicates the direct sum, and θ, ϕ are epimorphisms which respect the injection of $H_\lambda \to *_{\lambda\in\Lambda} H_\lambda$ and the projection $\oplus_{\lambda\in\Lambda} H_\lambda \to H_\lambda$.

2. The group d.g. near-ring. Let (R, S) be an arbitrary d.g. near-ring and G a multiplicative group. Let $(\bar{R}, S)$ be the upper faithful d.g. near-ring for (R, S) together with the natural homomorphism θ: $(\bar{R}, S) \twoheadrightarrow (R, S)$. Since $(\bar{R}, S)$ is faithful we can construct $(\bar{R}(G), SG)$. Let $I = \operatorname{Ker}\theta$ and $IG = \{ag;\ a \in I, g \in G\}$. Denote by J the ideal $\operatorname{Id}\langle IG\rangle$. By the remark above, J is the normal subgroup of $(\bar{R}(G), +)$ generated by

$$\bar{R}(G)IGSG = \Big\{\big(\sum r_i g_i\big)(ag)(sh), \big(\sum r_i g_i\big)(ag), (ag)(sh), ag;$$
$$\sum r_i g_i \in \bar{R}(G), a \in I, s \in S, g, h \in G\Big\}.$$

Using results about $\bar{R}(G)$ from [4], we have

$$\bar{R}(G)IGSG = \Big\{\big(\sum r_i g_i(ag), ag;\ \sum r_i g_i \in \bar{R}(G), a \in I, g \in G\Big\}$$

since I is an ideal of $\bar{R}$.

Definition 1. $(\bar{R}(G), SG)/J$ is called the group d.g. near-ring of (R, S) on G.

Without danger of confusion we may denote it $(R(G), SG)$, as we will see later that $SG + J/J$ is naturally isomorphic to SG. The following generalization of

Fröhlich's result—if (R, S) is a d.g. near-ring and H is an S group, then H is an (R, S) group provided $\operatorname{Ker} \pi \subseteq \operatorname{Ann}_{\operatorname{Fr}(S)} H$, where π is the natural homomorphism from the free d.g. near-ring $(\operatorname{Fr}(S), S)$ on S to (R, S)—is needed for our first result.

LEMMA 2. *Let ϕ: $(R, S) \twoheadrightarrow (T, U)$ be an epimorphism, and let H be an (R, S) group which is also a U group. Then H is a (T, U) group if* $\operatorname{Ker} \phi \subseteq \operatorname{Ann}_R H$.

PROOF. Let ψ be the representation of (R, S) on H. By the hypothesis, $\operatorname{Ker} \phi \subseteq \operatorname{Ker} \psi$. Hence ψ factors through ϕ giving a representation of (T, U) on H.

THEOREM 3. *$(\bar{R}(G)/J, +)$ is an (R, S) group.*

PROOF. Clearly $(\bar{R}(G)/J, +)$ is an $(\bar{R}, S)$ group. Let $\Sigma r_i g_i + J \in \bar{R}(G)/J$, $a \in I$. Then

$$\left(\sum r_i g_i + J\right)a = \left(\sum r_i g_i\right)a + J = J,$$

since $a \in I$ and $J \supseteq IG$. Therefore $I \subseteq \operatorname{Ann}_{\bar{R}}(\bar{R}(G)/J)$. Hence $(\bar{R}(G)/J, +)$ is an (R, S) group by Lemma 2.

We note that

$$(\bar{R}(G)/J, +) = Gp\left\langle (\bar{R}g + J)/J;\ g \in G \right\rangle,$$

where each $(\bar{R}g + J)/J$ is an additive subgroup of $\bar{R}(G)/J$. Moreover, the groups $\{(\bar{R}g + J)/J;\ g \in G\}$ can be considered as (R, S) groups in a natural way, as in Theorem 3.

We now look at the relationship of $(R(G), SG)$ to $(\underline{R}(G), \underline{S}G)$. Let $(\underline{R}, \underline{S})$ be the lower faithful d.g. near-ring for (R, S) together with the natural homomorphism $\underline{\theta}$: $(R, S) \twoheadrightarrow (\underline{R}, \underline{S})$. Then as before we can construct the group d.g. near-ring $(\underline{R}(G), \underline{S}G)$ which is a sub-d.g. near-ring of $(E(\underline{F}), \operatorname{End} \underline{F})$, where $\underline{F} = \operatorname{Fr}(Y, \underline{R}, \underline{S})$, the free $(\underline{R}, \underline{S})$ group on Y. Let $\bar{F} = \operatorname{Fr}(Y, \bar{R}, S)$ be the free $(\bar{R}, S)$ group on Y. Clearly, $(\bar{R}(G), SG)$ is a sub-d.g. near-ring of $(E(\bar{F}), \operatorname{End} \bar{F})$. By the freeness of $\bar{F}$ there exists a unique $(\bar{R}, S)$ homomorphism μ: $\bar{F} \twoheadrightarrow \underline{F}$ which extends the identity map on Y. Note that since $(\underline{R}, \underline{S})$ is a homomorphic image of $(\bar{R}, S)$ under $\theta\underline{\theta}$, $\underline{F}$ is an $(\bar{R}, S)$ group in a natural way. We have

$$((x, g)r)\mu = (x, g)(r\theta\underline{\theta}) \quad \text{for all } (x, g) \in Y, r \in \bar{R}.$$

LEMMA 4. *α: $SG \to \underline{S}G$ defined by $sg \to (s\theta\underline{\theta})g$ extends to a homomorphism from $(\bar{R}(G), SG)$ to $(\underline{R}(G), \underline{S}G)$.*

PROOF. α is certainly a semigroup homomorphism $SG \to \underline{S}G$, so we need only check that it extends to a group homomorphism $(\bar{R}(G), +) \to (\underline{R}(G), +)$, which we will also denote by α.

Let $r = \varepsilon_1 s_1 g_1 + \cdots + \varepsilon_n s_n g_n = 0$ in $\bar{R}(G)$, where $\varepsilon_i = \pm 1$, $s_i \in S$, $g_i \in G$. Then

$$r\alpha = \varepsilon_1(s_1\theta\underline{\theta})g_1 + \cdots + \varepsilon_n(s_n\theta\underline{\theta})g_n$$

has to be shown to be 0 in $\underline{R}(G)$. Since $\underline{F}$ is the free $(\underline{R}(G), \underline{S}G)$ group on $\{(x, 1);\ x \in X\}$, we need only show that $(x, 1)r = 0$ for all $x \in X$. But $(x, 1)r = 0$

for all $x \in X$ in $\bar{F}$. Hence

$$\begin{aligned} 0 &= ((x,1)r)\mu = ((x,1)(\varepsilon_1 s_1 g_1 + \cdots + \varepsilon_n s_n g_n))\mu \\ &= (\varepsilon_1(x, g_1)s_1 + \cdots + \varepsilon_n(x, g_n)s_n)\mu \\ &= (\varepsilon_1(x, g_1)s_1)\mu + \cdots + (\varepsilon_n(x, g_n)s_n)\mu \quad \text{since } \mu \text{ is a homomorphism} \\ &= \varepsilon_1(x, g_1)(s_1\theta\underline{\theta}) + \cdots + \varepsilon_n(x, g_n)(s_n\theta\underline{\theta}) \\ &= \varepsilon_1(x, 1)(s_1\theta\underline{\theta})g_1 + \cdots + \varepsilon_n(x, 1)(s_n\theta\underline{\theta})g_n \\ &= (x,1)(\varepsilon_1(s_1\theta\underline{\theta})g_1 + \cdots + \varepsilon_n(s_n\theta\underline{\theta})g_n) = (x,1)(r\alpha). \end{aligned}$$

This suffices to prove the result.

THEOREM 5. *$(\underline{R}(G), \underline{S}G)$ is a homomorphic image of $(\bar{R}(G), SG)/J$.*

PROOF. It suffices to show that $J \subseteq \operatorname{Ker}\alpha$. Let $ag \in IG$. Then $(ag)\alpha = (a\theta\underline{\theta})g = (0\underline{\theta})g = 0$ in $\underline{R}(G)$. Hence $IG \subseteq \operatorname{Ker}\alpha$, and so $J \subseteq \operatorname{Ker}\alpha$, since $J = \operatorname{Id}\langle IG \rangle$ and $\operatorname{Ker}\alpha$ is an ideal.

We thus have the following commutative diagram:

$$\begin{tikzcd} (\bar{R}(G), SG) \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\alpha"'] & (\bar{R}(G), SG)/J \arrow[dl, two heads, "\beta"] \\ (\underline{R}(G), \underline{S}G) & \end{tikzcd}$$

Here π is the natural homomorphism. Note that β is uniquely defined.

We now wish to show that $(R(G), SG)$ is an orthogonal sum of (R, S) groups $(Rg, +)$ each isomorphic to $(R, +)$.

THEOREM 6. *$(\bar{R}(G), SG)/J$ is an orthogonal sum of (R, S) groups $(Rg, +)$ each isomorphic to $(R, +)$.*

PROOF. Consider the following diagram:

$$\begin{tikzcd} \bar{R}g \arrow[r, "\alpha_g"] \arrow[d, "\theta_g"] & (\bar{R}(G), SG) \arrow[r, "\psi"] \arrow[d, "\pi"] & \bigoplus_{g\in G} \bar{R}g \arrow[r, "\beta_g"] \arrow[d, "\bar{\theta}"] & \bar{R}g \arrow[d, "\theta_g"] \\ Rg \arrow[r, "\delta_g"'] & (\bar{R}(G), SG)/J \arrow[r, "\phi"'] & \bigoplus_{g\in G} Rg \arrow[r, "\gamma_g"'] & Rg \end{tikzcd}$$

Rg is an (R, S) group, hence an $(\bar{R}, S)$ group, which is isomorphic to $(R, +)$ and whose elements are $\{rg \colon r \in R\}$. The maps θ_g, $\bar{\theta}$, $\underline{\theta_g}$ are the obvious $(\bar{R}, S)$ homomorphisms induced by θ. Note $\operatorname{Ker}\theta_g = Ig$, $\operatorname{Ker}\bar{\theta} = \bigoplus_{g\in G} Ig$. Finally π is the canonical homomorphism, α_g, β_g and ψ are the maps arising from the orthogonal sum properties of $\bar{R}(G)$, and γ_g are the usual projections. We wish to show the existence of homomorphisms δ_g, ϕ making the diagram commutative. Note that the right-hand square is commutative, as can be seen from the definitions of the maps. So $\beta_g\theta_g = \bar{\theta}\gamma_g$.

Consider $Ig \subseteq \bar{R}(G)$ for some $g \in G$. Then $(Ig)\psi \subseteq Ig \subseteq \bar{R}G \subseteq \oplus_{g\in G} \bar{R}g$, from the definition of ψ. Hence $(Ig)\psi \subseteq \operatorname{Ker}\bar{\theta}$. This holds for all $g \in G$. Thus $IG \subseteq \operatorname{Ker}\psi\bar{\theta}$. Since $\operatorname{Ker}\psi\bar{\theta}$ is an ideal, it follows that $J = \operatorname{Id}\langle IG\rangle \subseteq \operatorname{Ker}\psi\bar{\theta}$. So $\psi\bar{\theta}$ factors uniquely through π, i.e., there exists ϕ: $(\bar{R}(G), SG)/J \to \oplus_{g\in G} Rg$ such that $\pi\phi = \psi\bar{\theta}$ and ϕ is unique. So ϕ exists and the middle square is commutative. It follows that $\operatorname{Ker}\pi\phi \cap \bar{R}g = Ig$. This leads to the following result, which we state separately.

Lemma 7. *In $\bar{R}(G)$, $\bar{R}g \cap J = Ig$ for all $g \in G$.*

We return to the proof of Theorem 6. Consider $\alpha_g\pi$. By the definition of orthogonal sum, α_g is a monomorphism. So

$$\operatorname{Ker}\alpha_g\pi = \alpha_g^{-1}(\operatorname{Ker}\pi \cap \operatorname{Im}\alpha_g) = \alpha_g^{-1}(J \cap \bar{R}g) = Ig \subseteq \bar{R}g$$

by Lemma 7. But $\operatorname{Ker}\theta_g = Ig$. So there exists a unique monomorphism δ_g: $Rg \to (\bar{R}(G), SG)/J$ such that $\theta_g\delta_g = \alpha_g\pi$. In particular, δ_g: $RG \to \bar{R}g + J/J$. Also, ϕ: $\bar{R}g + J/J \to Rg \subseteq \oplus_{g\in G} Rg$. The complete diagram is commutative and $(\bar{R}(G), SG)/J$ is an orthogonal sum of the groups $\bar{R}g + J/J$, each of which is isomorphic to $(R, +)$.

Corollary 8. *$(\bar{R}g + J)/J \cong (Rg, +) \cong (R, +)$ as (R, S) group for each $g \in G$.*

Note that Lemma 7 implies that $J \cap SG$ is trivial and, hence, that $SG + J/J \cong SG$ as a semigroup. So we can write $(R(G), SG)$ for $(\bar{R}(G), SG)/J$, and we can identify $SG + J/J$ with SG, $\bar{R}g + J/J$ with Rg for each $g \in G$.

We note that for any group d.g. near-ring $(R(G), SG)$, the subnear-ring $(R1_G, S1_G)$ is naturally isomorphic to (R, S). Since a sub-d.g. near-ring of a faithful d.g. near-ring is faithful, it follows that if (R, S) is not faithful, then neither is $(R(G), SG)$ for any group G. We do have a faithful d.g. near-ring with a projection on to $(R(G), SG)$, namely π: $(\bar{R}(G), SG) \to (R(G), SG)$. We also have a faithful d.g. near-ring which is a homomorphic image of $(R(G), SG)$, namely β: $(R(G), SG) \to (\underline{R}(G), \underline{S}G)$. We now relate these to the upper and lower faithful d.g. near-rings for (R, S).

Theorem 9. *Let ϕ: $(U, SG) \to (R(G), SG)$ be the upper faithful d.g. near-ring for $(R(G), SG)$. Then (U, SG) is an orthogonal sum of the $(\bar{R}, S)$ groups $\{(\bar{R}g, +);\ g \in G\}$ and the canonical homomorphism ψ: $(\bar{R}(G), SG) \to (U, SG)$ such that $\psi\phi = \pi$ respects the orthogonal sum structure.*

In Fröhlich's notation, ψ is an orthogonal homomorphism.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
(\bar{R}(G), SG) \arrow[r, "\pi"] \arrow[d, "\psi"] & (R(G), SG) \\
(U, SG) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where all the maps are epimorphisms and restrict to the identity on SG. Denote by (Tg, Sg) the sub-$(\bar{R}, S)$ group of $(U, +)$ generated by $(Sg)\psi$, which we identify with Sg. Then $(T1_G, S1_G)$ is a d.g. near-ring. It is faithful, as it is a sub-d.g. near-ring of a faithful d.g. near-ring. From the properties of upper faithful d.g. near-rings, it follows that $(T1_G, S1_G) \cong (\bar{R}1_G, S1_G)$. Hence, $(Tg, Sg) = (T1_G, S1_G)g \cong (\bar{R}g, Sg)$. Further, ψ: $\bar{R}g \to Tg$ and is the identity on $Sg \to Sg$. Thus $\operatorname{Ker} \psi \cap \bar{R}g = \{0\}$. This leads to an embedding of $(\bar{R}g, Sg)$ in (U, SG) for each $g \in G$.

We now need projections of (U, SG) onto $(\bar{R}g, Sg)$ for each $g \in G$. With a change in notation, we use the proof of Theorem 6 to obtain the following commutative diagram:

$$\begin{tikzcd} & (\bar{R}(G), SG) \arrow[r, "\beta_g"] \arrow[dl, "\psi"'] \arrow[d, "\pi"] & \bar{R}g \arrow[d, "\theta_g"] \\ (U, SG) \arrow[r, "\phi"] & (R(G), SG) \arrow[r, "\gamma_g"] & Rg \end{tikzcd}$$

For $g = 1_G$ we have a map $\phi\gamma_1$: $(U, SG) \to R1_G$, and $\phi\gamma_1$ maps $(T1_G, S1_G) \to (R1_G, S1_G)$. As before, it follows that $\phi\gamma_1$ restricted to $T1_G$ factors through θ_1: $\bar{R}1_G \to R1_G$, using the properties of upper faithful d.g. near-rings. Now right multiplication by g maps $T1_G$ to Tg and $R1_G$ to Rg. Hence, $\phi\gamma_g$ always factors through θ_g, when restricted to Tg. This finishes the proof of the result.

THEOREM 10. *Let* $\underline{\phi}$: $(R(G), SG) \to (\underline{U}, \underline{SG})$ *be the lower faithful d.g. near-ring for* $(R(G), SG)$. *Then* $(\underline{U}, \underline{SG})$ *is an orthogonal sum of the* $(\underline{R}, \underline{S})$ *groups* $\{(\underline{R}g, +); g \in G\}$ *and the canonical homomorphism* $\underline{\psi}$: $(\underline{U}, \underline{SG}) \to (\underline{R}(G), \underline{S}G)$ *such that* $\beta = \underline{\phi}\underline{\psi}$ *respects the orthogonal sum structure.*

PROOF. The proof parallels that of Theorem 9 fairly closely. So we will only give an outline. Consider the following commutative diagram:

$$\begin{tikzcd} (R(G), SG) \arrow[r, "\beta"] \arrow[d, "\underline{\phi}"'] & (\underline{R}(G), \underline{S}G) \\ (\underline{U}, \underline{SG}) \arrow[ur, "\underline{\psi}"'] & \end{tikzcd}$$

Denote by $(\underline{T}g, \underline{S}g)$ the sub-$(\underline{R}, \underline{S})$ group of $(\underline{U}, +)$ generated by $(Sg)\underline{\phi}$. Considering first $\underline{T}1_G$, we see as before that $(\underline{T}1_G, \underline{S}1_G) \cong (\underline{R}1_G, \underline{S}1_G)$. This justifies the assumption made in the statement of the theorem that $(SG)\underline{\phi} = \underline{S}G$. Using right multiplication by g gives us $(\underline{T}g, \underline{S}g) \cong (\underline{R}g, \underline{S}g)$, and we have the embedding of $(\underline{R}g, Sg)$ in $(\underline{U}, \underline{SG})$ for each $g \in G$.

For the second part, we have an easier situation. From above, we know that $\underline{\psi}$ respects the embeddings. Since $(\underline{R}(G), \underline{S}G)$ is an orthogonal sum of $\{(\underline{R}g, \underline{S}g); g \in G\}$, and $(\underline{U}, \underline{SG})$ is mapped onto it by a homomorphism $\underline{\psi}$ respecting the embeddings, it follows that $(\underline{U}, \underline{SG})$ is an orthogonal sum of $\{(\underline{R}g, \underline{S}g); g \in G\}$, and $\underline{\psi}$ respects the orthogonal sum structure.

The next theorem follows immediately.

THEOREM 11. *If* $(\underline{R}(G), \underline{S}G)$ *is the free* $(\underline{R}, \underline{S})$ *sum of* $\{(\underline{R}g, \underline{S}g); g \in G\}$, *then* β: $(R(G), SG) \to (\underline{R}(G), \underline{S}G)$ *is the lower faithful d.g. near-ring for* $(R(G), SG)$.

If (U, SG) is the free $(\bar{R}, S)$ sum of $\{(\bar{R}g, Sg);\ g \in G\}$, then π: $(\bar{R}(G), SG) \rightarrow (\bar{R}(G), SG)$ is the upper faithful d.g. near-ring for $(R(G), SG)$, and $(\bar{R}(G), SG)$ is the free $(\bar{R}, S)$ sum of $\{(\bar{R}g, Sg);\ g \in G\}$.

The concrete determination of the upper and lower faithful d.g. near-rings for a given d.g. near-ring is difficult and involves a good deal of group theory in the form of group presentations in all cases covered so far. The case of lower faithful d.g. near-rings has been treated in [**5**] and that of upper faithful d.g. near-rings in [**6**]. In [**6**] the near-rings considered in detail are the zero near-rings on the finite dihedral groups. The smallest example of a group d.g. near-ring, namely $(R(G), SG)$ for $(R, +)$ the dihedral group of order 6 and G the cyclic group of 2, needs a very sophisticated group theoretic treatment, as anyone who consults [**5** or **6**] can see. So we are not in a position to give details here. But we hope to examine this situation in some detail in a later paper.

There are some interesting questions which arise from the last two theorems. When is $(\bar{R}(G), SG)$ the upper faithful d.g. near-ring for $(R(G), SG)$? And when is $(\underline{R}(G), \underline{S}G)$ the lower faithful d.g. near-ring for $(R(G), SG)$? These seem to be hard questions whose answer will depend on a detailed knowledge of the structure of the corresponding groups. This is also true of the problem of giving an "interval" characterization of $(R(G), SG)$, that is, one that does not involve going through $(\bar{R}, S)$.

Finally, a comment about group near-rings for arbitrary near-rings: The structure of a group near-ring is closely related to the free near-ring-module product. In the case of a zero-symmetric near-ring, this product exists, as general theorems about free products in varieties assure us. But detailed structural results are only emerging now, and they lead to a very complicated structure. Again it is hoped that these results will be followed up at a later stage.

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