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THE SPECTRAL DECOMPOSITION OF A PRODUCT OF AUTOMORPHIC FORMS

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ABSTRACT. The spectral theory of Roelke and Selberg provides a decomposition of the space of square integrable automorphic forms for the group $SL(2)$ in terms of eigenfunctions of the non-Euclidean Laplacian and of the Hecke operators. The main result of the paper uses the Roelke-Selberg theory to give an interpretation of the L-functions of Rankin type as "multiplicity factors" in the decomposition of the product of a nonholomorphic Eisenstein series and a cusp form.

Let H denote the upper half plane and Γ the modular group. As is well known the spectral decomposition of the space $L^2(\Gamma\backslash H)$ contains a discrete and a continuous part [**1**, p. 62]. It is also known that a basis for the discrete part can be chosen to consist of eigenfunctions for all the Hecke operators including the Laplacian. The integral kernel describing the continuous part involves real analytic Eisenstein series and these are also eigenfunctions for the Hecke operators. The aim of this note is to give an explicit decomposition for the product of a function in the discrete part with one in the continuous part.

Let $L(s, \pi_f) = \pi^{-s}\Gamma((s+r)/2)\Gamma((s-r)/2)\prod_p(1 - a(p)p^{-s} + p^{-2s})^{-1}$ be the Euler product corresponding to a normalized eigenfunction f of all the Hecke operators, i.e.

$$T_p \cdot f = a(p)\cdot f \quad \text{and} \quad \Delta\cdot f = \frac{1-r^2}{2}\cdot f$$

where $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplacian; we are also assuming $f(-\bar{z}) = f(z)$. Let $\{f_j\}_{j\in\mathbf{N}}$ be an orthogonal basis for the discrete part of the spectrum of $L^2(\Gamma\backslash H)$ consisting of eigenfunctions for all the Hecke operators. Let $E(z, s)$ be the real analytic Eisenstein series for Γ. If π_j is the automorphic representation corresponding to f_j, we denote by $L(s, \pi_f \times \pi_j)$ the Rankin convolution of the Euler products $L(s, \pi_f)$ and $L(s, \pi_j)$. Let $\langle\ ,\ \rangle$ denote the Petersson inner product on $L^2(\Gamma\backslash H)$ and put $\|\pi_j\|^2 = \langle f_j, f_j\rangle$ and $f_j^0 = \|\pi_j\|^{-2}f_j$. The Riemann zeta function is denoted by $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Our main result is the following

THEOREM. *The spectral decomposition of the product of a cusp form f in $L^2(\Gamma\backslash H)$ which is an eigenfunction of all the Hecke operators with the real analytic Eisenstein*

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series for Γ *is given by*

$$f(z)E(z,s) = \sum_j L\left(\frac{1+s}{2}, \pi_f \times \pi_j\right) f_j^0(z)$$
$$+ \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{L\left(\frac{1+s+\bar{s}'}{2}, \pi_f\right) L\left(\frac{1+s-\bar{s}'}{2}, \pi_f\right)}{\Lambda(1+s)\Lambda(1+\bar{s}')} E(z,s')\,dr$$
$$(s' = \tfrac{1}{2} + ir)$$

where $L(s, \pi_f \times \pi_j)$ *is the Rankin convolution of* $L(s, \pi_f)$ *and* $L(s, \pi_j)$.

PROOF. The spectral decomposition of an arbitrary function g in $L^2(\Gamma\backslash H)$ is given by [**1**, p. 62]

$$g(z) = \sum_j \langle g, f_m \rangle f_j^0(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \langle g, E(z,s') \rangle E(z,s')\,dr \qquad (s' = \tfrac{1}{2} + ir),$$

where $\langle g, f_j \rangle = \int_{\Gamma\backslash H} g(z)\cdot\overline{f_j(z)}\,d\Omega$ with $d\Omega = y^{-2}\,dx\,dy$ is the Γ-invariant measure on H. The Eisenstein series for Γ is defined by

$$E(z,s) = \sum_{\sigma\in\Gamma_\infty/\Gamma} \operatorname{Im}\sigma(z)^{(1+s)/2} \qquad (\Gamma_\infty = \{\sigma\in\Gamma : \sigma(\infty) = \infty\}).$$

If we apply the above formula to $g(z) = f(z)E(z,s)$, which is a function in $L^2(\Gamma\backslash H)$, we obtain for the coefficients of the discrete part:

$$\left\langle f(z)E(z,s), f_j(z)\right\rangle = \int_{\Gamma\backslash H} f(z)\overline{f_j(z)}\,E(z,s)\,d\Omega$$
$$= L\left(\frac{1+s}{2}, \pi_f \times \pi_j\right);$$

this is by definition the Rankin convolution of $L(s, \pi_f)$ and $L(s, \pi_j)$. It is an Euler product whose explicit form has been calculated in [**2**, p. 153]. Therefore the proof of the theorem reduces to calculating $\langle f(z)E(z,s), E(z,s')\rangle$, the Petersson inner product, and this we now proceed to do. Using the Fourier expansions

$$E(z,s) = y^{(1+s)/2} + \frac{\Lambda(s)}{\Lambda(1+s)} y^{(1-s)/2}$$
$$+ \sum_{m\neq 0} \frac{2}{\Lambda(1+s)} \frac{\sigma_s(|m|)}{|m|^{s/2}} y^{1/2} K_{s/2}(2\pi|m|y) e^{2\pi i m x}$$

and

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_r(2\pi|n|y) e^{2\pi i n x},$$

we obtain

$$\langle fE(z,s), E(z,s')\rangle = \int_{\Gamma\backslash H} f(z)E(z,s)\overline{E(z,s')}\,d\Omega$$

$$= \int_{|x|\le 1/2} f(z)\overline{E(z,s')}\,y^{(1+s)/2}\,d\Omega \qquad \left(\int_{|x|\le 1/2} = \int_{-\infty}^{\infty}\int_{-1/2}^{1/2}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-1/2}^{1/2}\left(\sum_{m\neq 0}(m)y^{1/2}K_r(2\pi|m|y)e^{2\pi imx}\right)$$

$$\times\left(\overline{y^{(1+s')/2} + \frac{\Lambda(s')}{\Lambda(1+s')}y^{(1-s')/2}}\right.$$

$$\left.\overline{+\sum_{n\neq 0}\frac{2}{\Lambda(1+s')}\frac{\sigma_{s'}(|m|)}{|n|^{s'/2}}y^{1/2}K_{s'/2}(2\pi|n|y)e^{2\pi inx}}\right)y^{(1+s)/2}\,d\Omega$$

$$= \int_{-\infty}^{\infty}\sum_{n\neq 0}a(n)y^{1/2}K_r(2\pi|n|y)\frac{2}{\Lambda(1+s')}\frac{\sigma_{s'}(|n|)y^{1/2}}{|n|^{s'/2}}K_{s'/2}(2\pi|n|y)y^{(1+s)/2}\,d\Omega$$

$$= \sum_{n\neq 0}a(n)\overline{\frac{2\sigma_{s'}(|n|)}{\Lambda(1+s')|n|^{s'/2}}}\int_{-\infty}^{\infty}y^{(s-1)/2}K_r(2\pi|n|y)K_{\bar{s}'/2}(2\pi|n|y)\,dy.$$

Letting $c(|n|) = 2\sigma_{s'}(|n|)/\Lambda(1+s')|n|^{s'/2}$, we obtain that the above is equal to

$$\sum_{n\neq 0}a(n)\overline{c(|n|)}\cdot\frac{(2\pi|n|)^{-(1+s)/2}}{2^{(1-s)/2+2}\Gamma\left(1-\left(\frac{1-s}{2}\right)\right)}\Gamma\left(\frac{1-\frac{1-s}{2}+r+\frac{\overline{s'}}{2}}{2}\right)$$

$$\times\Gamma\left(\frac{1-\frac{1-s}{2}+r-\frac{\overline{s'}}{2}}{2}\right)\Gamma\left(\frac{1-\frac{1-s}{2}-r+\frac{\overline{s'}}{2}}{2}\right)\Gamma\left(\frac{1-\frac{1-s}{2}-r-\frac{\overline{s'}}{2}}{2}\right)$$

$$= 2^{-3}\frac{\Gamma\dfrac{\left(\frac{s+1}{2}+r+\frac{\overline{s'}}{2}\right)}{2}\Gamma\dfrac{\left(\frac{s+1}{2}+r-\frac{\overline{s'}}{2}\right)}{2}}{\Gamma\left(\frac{s+1}{2}\right)}$$

$$\times\frac{\Gamma\dfrac{\left(\frac{s+1}{2}-r+\frac{\overline{s'}}{2}\right)}{2}\Gamma\dfrac{\left(\frac{s+1}{2}-r-\frac{\overline{s'}}{2}\right)}{2}\pi^{-(s+1)/2}}{\Gamma\left(\frac{s+1}{2}\right)}$$

$$\times\sum_{n\neq 0}a(n)\overline{c(|n|)}\,|n|^{-(1+s)/2}$$

The assumption $f(-\bar{z}) = f(z)$ implies that

$$\sum_{n\neq 0} a(n)\,\overline{c(|n|)}\,|n|^{-(1+s)/2} = 2\sum_{n=1}^{\infty} a(n)\,\overline{c(n)}\,n^{-(1+s)/2}$$

$$= 2\sum_{n=1}^{\infty} a(n)\frac{2}{\Lambda(1+s')}\,\frac{\sigma_{\bar{s}'}(n)}{|n|^{\bar{s}'/2}}\cdot\frac{1}{n^{(1+s)/2}}$$

$$= \frac{4}{\Lambda(1+s')}\sum_{n=1}^{\infty}\frac{a(n)\sigma_{s^*}(n)}{n^{(1+s+s^*)/2}}$$

$$= \frac{4}{\Lambda(1+s^*)}\prod_p\left(\sum_{v=0}^{\infty} a(p^v)\sigma_{s^*}(p^v)p^{-v((1+s+s^*)/2)}\right),$$

where we have put $s^* = \overline{s'}$.

Now recall that the coefficients $a(p^v)$ satisfy

$$\frac{1}{1-a(p)T+T^2} = \sum_{v=0}^{\infty} a(p^v)T^v;$$

hence

$$a(p^v)\frac{p^{s^*(v+1)}-1}{p^{s^*}-1}p^{-v((1+s+s^*)/2)}$$
$$= \frac{a(p^v)p^{s^*}p^{-v((1+s+s^*)/2)} - a(p^v)p^{-v((1+s+s^*)/2)}}{p^{s^*}-1}.$$

Therefore

$$\sum_{v=0}^{\infty} a(p^v)\sigma_{s^*}(p^v)p^{-v((1+s+s^*)/2)}$$
$$= \frac{1-p^{-(1+s)}}{\{1-a(p)p^{-((1+s+s^*)/2)}+p^{-(1+s+s^*)}\}\{1-a(p)p^{-((1+s-s^*)/2)}+p^{-(1+s-s^*)}\}}.$$

Letting $\zeta(s,\pi_f) = \prod_p(1-a(p)p^{-s}+p^{-2s})^{-1}$, we can now write

$$\sum_{n\neq 0} a(n)\,\overline{c(|n|)}\,|n|^{-(1+s)/2}$$
$$= \frac{4}{\Lambda(1+s^*)}\cdot\frac{1}{\zeta(1+s)}\zeta\left(\frac{1+s-s^*}{2},\pi_f\right)\zeta\left(\frac{1+s+s^*}{2},\pi_f\right).$$

Putting the appropriate Γ-factors we finally obtain

$$\langle f(z)E(z,s), E(z,s')\rangle = \frac{1}{2}\,\frac{L\left(\dfrac{1+s-s^*}{2}, \pi_f\right)L\left(\dfrac{1+s+s^*}{2}, \pi_f\right)}{\Lambda(1+s)\Lambda(1+s^*)} \qquad (s^* = \overline{s'}).$$

And this is what we wanted to show.

References

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