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A MATRIX INVERSE

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ABSTRACT. George Andrews has demonstrated that the Bailey transform is equivalent to the inversion of an infinite-dimensional matrix whose entires are rational functions in q. We generalize this inversion by introducing an extra parameter which brings much greater symmetry.

Let $A = \{A_{nk}\}_{n,k=0}^{\infty}$ be an infinite-dimensional lower triangular matrix; $k > n$ implies that $A_{nk} = 0$. We say that A has an inverse, written $A^{-1} = \{A_{nk}^{-1}\}$, if

$$\sum_{k=m}^{n} A_{nk} A_{km}^{-1} = \delta_{nm},$$

for all nonnegative n and m, $m \leqslant n$. The inversion of such matrices when the entries are rational functions in q plays an important role in q-series identities.

Andrews [**1**] has shown that the Bailey transform [**2**, **3**] used to prove and generalize the Rogers-Ramanujan identities is equivalent to the following matrix inversion.

Let $B = \{B_{nk}\}_{n,k=0}^{\infty}$ where

$$B_{nk} = \frac{1}{(q)_{n-k}(aq)_{n+k}},$$

$(a)_{\infty} = \prod_{i=0}^{\infty}(1 - aq^i)$, $(a)_m = (a)_{\infty}/(aq^m)_{\infty}$. Then $B^{-1} = \{B_{nk}^{-1}\}$ where

$$B_{nk}^{-1} = \frac{(1 - aq^{2n})(a)_{k+n}(-1)^{n-k} q^{\binom{n-k}{2}}}{(1-a)(q)_{n-k}}.$$

More recently, Gessel and Stanton proved a number of q-series identities using this same inversion (Theorem 1.2 of [**5**]).

Andrews' inversion is a special case of a far more appealing result.

THEOREM. *Let* $D = \{D_{nk}(a, b)\}_{n,k=0}^{\infty}$ *where*

$$D_{nk}(a,b) = \frac{(1 - aq^{2k})(b)_{k+n}(ba^{-1})_{n-k}(ba^{-1})^k}{(1-a)(aq)_{k+n}(q)_{n-k}}.$$

Then $D^{-1} = \{D_{nk}(b, a)\}_{n,k=0}^{\infty}$.

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For each $n = k \geqslant 0$, if the kth column of $D(a, b)$ is divided by

$$\frac{1 - aq^{2k}}{1 - a}(ba^{-1})^k$$

and the nth row of $D(b, a)$ is multiplied by

$$\frac{1 - aq^{2n}}{1 - a}(ba^{-1})^n,$$

then we get two matrices which are inverses to each other and which give Andrews' inversion when $b = 0$. The inversion in this theorem is equivalent to the transform given by the author in [**4**].

PROOF OF THEOREM. We compute the inner product of the nth row of D with the mth column of D^{-1}. If $m > n$, then the inner product is trivially zero and so we assume that $m \leqslant n$. (a)

(1)

$$\sum_{k=m}^{n} D_{nk}(a, b)D_{km}(b, a)$$

$$= \sum_{k=m}^{n} \frac{(1 - aq^{2k})(b)_{k+n}(ba^{-1})_{n-k}(ba^{-1})^k(1 - bq^{2m})(a)_{m+k}(ab^{-1})_{k-m}(ab^{-1})^m}{(1 - a)(aq)_{k+n}(q)_{n-k}(1 - b)(bq)_{m+k}(q)_{k-m}}.$$

Setting $k = j + m$ and pulling out of the summation those factors which do not depend on j yields

$$\sum D_{nk}(a, b)D_{km}(b, a) = \frac{(1 - aq^{2m})(1 - bq^{2m})(b)_{m+n}(ba^{-1})_{n-m}(a)_{2m}}{(1 - a)(1 - b)(aq)_{m+n}(q)_{n-m}(bq)_{2m}} \tag{2}$$
$$\cdot \sum_{j=0}^{n-m} \frac{(1 - aq^{2m+2j})(bq^{m+n})_j(aq^{2m})_j(ab^{-1})_j(q^{n-m-j+1})_j(ba^{-1})^j}{(1 - aq^{2m})(ba^{-1}q^{n-m-j})_j(aq^{n+m+1})_j(bq^{2m+1})_j(q)_j}.$$

If $m = n$, the summation reduces to a single term equal to one and the entire right side of (2) is easily seen to equal one. If $m < n$, it is sufficient to show that the summation on the right of (2) is zero. By the limiting form of Jackson's theorem (equation A5 in [**4**]), this summation is equal to

$$\frac{(aq^{2m+1})_{n-m}(q^{m-n+1})_{n-m}}{(ab^{-1}q^{m-n+1})_{n-m}(bq^{2m+1})_{n-m}},$$

which is zero since $m < n$.

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