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ITERATION OF HOLOMORPHIC MAPS OF THE UNIT BALL INTO ITSELF

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Dedicated to Professor Mitsuru Ozawa on the occasion of his 60*th birthday*

ABSTRACT. Let Ω be a plane disc and let f be a holomorphic map of Ω into itself. It is known that the iterates f_n of f converge to a constant $\zeta \in \bar{\Omega}$ as $n \to \infty$ unless f is a conformal map of Ω onto itself. In the present paper it is shown that a more complicated statement of this kind is true in the unit ball of $\mathbf{C}^N$.

1. Let Ω be a plane disc and let f be a holomorphic map of Ω into itself. It is known that the iterates f_n of f converge to a constant $\zeta \in \bar{\Omega}$ as $n \to \infty$ unless f is a conformal map of Ω onto itself [**3**, pp. 131–134]. In the present paper we study the action of the iterates of holomorphic maps of the unit ball into itself.

Let B_N be the unit ball of $\mathbf{C}^N$ and let F be a holomorphic map of B_N into itself. The iterates F_n of F are defined by

$$F_0(z) = z, \quad F_{n+1}(z) = F(F_n(z)) \qquad (n = 0, 1, \ldots).$$

Let r be a positive integer with $1 \leqslant r \leqslant N$. Corresponding to the orthogonal direct sum decomposition $\mathbf{C}^N = \mathbf{C}^r \oplus \mathbf{C}^{N-r}$, each $z \in \mathbf{C}^N$ decomposes into $z = z' + z''$, where $z' \in \mathbf{C}^r$, $z'' \in \mathbf{C}^{N-r}$. Accordingly, for a map $\Phi = (\phi_1, \ldots, \phi_N)$ from B_N into $\mathbf{C}^N$, we write $\Phi = \Phi' + \Phi''$, where $\Phi' = (\phi_1, \ldots, \phi_r)$ and $\Phi'' = (\phi_{r+1}, \ldots, \phi_N)$.

THEOREM. *Let F be a holomorphic map of B_N into itself. If F is not an automorphism of B_N, then either*

(i) *the iterates F_n of F converge to a constant map as $n \to \infty$, uniformly on every compact subset of B_N, or*

(ii) *there exist a subsequence $\{F_{n_\nu}\}$ of $\{F_n\}$, an automorphism T of B_N and a positive integer r with $1 \leqslant r < N$, such that $T \circ F_{n_\nu} \circ T^{-1}$ converge as $\nu \to \infty$, uniformly on every compact subset of B_N, to a holomorphic map Φ of B_N into itself having the following properties*: (a) $\Phi'(z') = z'$ *for* $z' \in B_r$ *and* (b) $\Phi''(z) = 0''$ *for* $z \in B_N$.

In case (i), *if* $\zeta = \lim_{n \to \infty} F_n(z)$, $z \in B_N$, *is a point in* B_N, *then* ζ *is the fixed point of* F.

In case (ii) *the restriction* $\tilde{F}'(z')$ *of* $\tilde{F} = T \circ F \circ T^{-1}$ *to* B_r *is an automorphism of* B_r.

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There is a simple example which belongs to case (ii). We define the map $F = (f_1, f_2)$ by

$$f_1(z) = -z_1 + az_2^2, \quad f_2(z) = bz_2, \qquad z = (z_1, z_2),$$

where $2|a| + |a|^2 + |b|^2 < 1$. Then F is a holomorphic map of B_2 into itself and F_{2k} converge as $k \to \infty$ to the map

$$\Phi(z) = \left(z_1 - \frac{a}{1+b^2} z_2^2, 0\right),$$

and F_{2k+1} converge as $k \to \infty$ to the map

$$\psi(z) = \left(-z_1 + \frac{a}{1+b^2} z_2^2, 0\right).$$

Finally we note that there is a holomorphic map F of the unit polydisc U^N in $\mathbf{C}^N$ into itself such that the iterates F_n converge to a nonconstant map from U^N into ∂U^N. A simple example is

$$F(z) = \left(\tfrac{2}{3}z_1 + \tfrac{1}{3}, z_2, \ldots, z_N\right), \qquad z = (z_1, \ldots, z_N).$$

2. We begin with an elementary lemma.

LEMMA. *Let Ψ be a holomorphic map from B_N into $\overline{B}_N$. If Ψ is not constant, then $\Psi(B_N) \subset B_N$.*

PROOF. Suppose that $\Psi = (\psi_1, \ldots, \psi_N)$ maps one point in B_N to a point w in the boundary of B_N. We may assume that $w = (1, 0, \ldots, 0)$. Then the maximum principle shows that $\psi_1(z) \equiv 1$, and so that $\psi_j(z) \equiv 0$, $j = 2, \ldots, N$. Thus Ψ must be constant.

Let Ψ be a holomorphic map from B_N into $\mathbf{C}^N$. We denote by $A_\Psi(z)$ the matrix (a_{ij}),

$$a_{ij} = \frac{\partial \psi_i}{\partial z_j}(z) \quad (1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N)$$

where $\Psi = (\psi_1, \ldots, \psi_N)$ and $z = (z_1, \ldots, z_N)$.

3. We now turn to the proof of the theorem. Let F be a holomorphic map of B_N into itself and let F_n be its nth iterate. Then the sequence $\{F_n\}$ is normal in B_N.

Case 1. Every convergent subsequence of $\{F_n\}$ converges to a constant map uniformly on every compact subset of B_N.

Case 1.1. There exists a subsequence $\{F_{n_v}\}$ converging to a constant map which maps B_N into a point ζ in B_N.

We have

$$F(\zeta) = \lim_{v \to \infty} F\left(F_{n_v}(z)\right) = \lim_{v \to \infty} F_{n_v}(F(z)) = \zeta$$

and, hence, for any convergent subsequence $\{F_{m_v}\}$ we have

$$\lim_{v \to \infty} F_{m_v}(z) = \lim_{v \to \infty} F_{m_v}(\zeta) = \zeta \qquad (z \in B_N).$$

Thus, in this case, $\{F_n\}$ converges to the constant ζ and ζ is the fixed point of F.

Case 1.2. Every convergent subsequence of $\{F_n\}$ converges to a constant map which maps B_N into a point in the boundary of B_N.

In this case we have $\|F_n(z^*)\| \to 1$ as $n \to \infty$, where z^* is a point in B_N. Hence there exists a subsequence $\{F_{n_v}\}$ such that

$$\|F_{n_v}(z^*)\| < \|F_{n_v+1}(z^*)\| \qquad (v = 1, 2, \ldots).$$

We may assume that $\{F_{n_v}(z)\}$, $z \in B_N$, converges to the point $e_1 = (1, 0, \ldots, 0)$. Put $a_v = F_{n_v}(z^*)$ $(v = 1, 2, \ldots)$. Then

$$\|a_v\| < \|F(a_v)\| \qquad (v = 1, 2, \ldots),$$

$$\lim_{v \to \infty} a_v = e_1, \quad \lim_{v \to \infty} F(a_v) = \lim_{v \to \infty} F_{n_v}(F(z^*)) = e_1.$$

Hence, letting $v \to \infty$ in the inequality

$$\frac{|1 - \langle F(a_v), F(z)\rangle|^2}{1 - \|F(z)\|^2} \leqslant \frac{1 - \|F(a_v)\|^2}{1 - \|a_v\|^2} \frac{|1 - \langle a_v, z\rangle|^2}{1 - \|z\|^2} \qquad (z \in B_N)$$

[**2**, p.163] we have

$$\frac{|1 - f_1(z)|^2}{1 - \|F(z)\|^2} \leqslant \frac{|1 - z_1|^2}{1 - \|z\|^2} \qquad (z \in B_N),$$

where $F = (f_1, \ldots, f_N)$ and $z = (z_1, \ldots, z_N)$. Then writing $F_n = (f_1^{(n)}, \ldots, f_N^{(n)})$ we obtain

$$\frac{|1 - f_1^{(n)}(z)|^2}{1 - \|F_n(z)\|^2} \leqslant \frac{|1 - z_1|^2}{1 - \|z\|^2} \qquad (z \in B_N).$$

Since $\|F_n(z)\| \to 1$ as $n \to \infty$, it follows from this inequality that $f_1^{(n)}(z) \to 1$ as $n \to \infty$. This shows that $F_n(z) \to e_1$ as $n \to \infty$.

Case 2. There exists a subsequence $\{F_{n_v}\}$ which converges to a nonconstant map Ψ, uniformly on every compact subset of B_N.

There is a point z^* in B_N with rank $A_\Psi(z^*) > 0$. We may assume that

$$\operatorname{rank} A_\Psi(z^*) = \sup_{\Phi \in \mathfrak{L}} \sup_{z \in B_N} \operatorname{rank} A_\Phi(z) \equiv r \leqslant N,$$

where $\mathfrak{L}$ is the family of all limit maps of convergent subsequences of $\{F_n\}$. Further we may assume that $\{F_{n_{v+1}-n_v}\}$ also converges. Let Φ be its limit map. Since $\Psi(B_N) \subset B_N$ (it follows from the lemma), we have

$$\Phi(\Psi(z)) = \lim_{v \to \infty} F_{n_{v+1}-n_v}\big(F_{n_v}(z)\big) = \Psi(z) \qquad (z \in B_N).$$

Putting $w^* = \Psi(z^*)$ we have $\Phi(w^*) = w^*$. Regarding A_Ψ and A_Φ as linear operators, rather than matrices, the relation $\Phi \circ \Psi = \Psi$ shows that $A_\Phi(w^*) \circ A_\Psi(z^*) = A_\Psi(z^*)$. This shows that the restriction of $A_\Phi(w^*)$ to the r-dimensional space which is the range of $A_\Psi(z^*)$ is the identity. Hence, by the definition of r, it follows that rank $A_\Phi(w^*) = r$.

Now we may assume that $\Phi(0) = 0$ and $A_\Phi(0) = \left(\begin{smallmatrix} I & X \\ Y & Z \end{smallmatrix}\right)$, rank $A_\Phi(0) = r$, by considering the map $T \circ F \circ T^{-1}$ in place of F, where I is the identity matrix of order r and T is a suitable automorphism of B_N.

We shall show that Φ has properties (a) and (b). Since $\Phi'(z')$ is a holomorphic map of B_r into itself with $\Phi'(0') = 0'$ and since $A_{\Phi'}(0') = I$, it follows by Cartan's uniqueness theorem that

$$\Phi'(z') = z' \qquad (z' \in B_r). \tag{1}$$

Further, from the inequality

$$\|z'\|^2 + \|\Phi''(z')\|^2 = \|\Phi(z')\|^2 < 1 \qquad (z' \in B_r)$$

we have that $\|\Phi''(z')\| \to 0$ as $\|z'\| \to 1$ (where $\| \ \|$ denotes the euclidean norm), and so

$$\Phi''(z') = 0'' \quad (z' \in B_r). \tag{2}$$

On the other hand, since rank $A_\Phi(z) = r$ in a neighborhood U of 0, there exist functions $H_j(z')$, $j = r+1, \dots, N$, such that $H_j(z')$ are holomorphic in a neighborhood U' of $0'$ and

$$\Phi''(z) = \big(H_{r+1}(\Phi'(z)), \dots, H_N(\Phi'(z))\big) \qquad (z \in U). \tag{3}$$

(For instance, this follows from Theorem 5 in Chapter I, §B of [**1**].) From (1), (2) and (3) we see that

$$\begin{aligned}(H_{r+1}(z'), \dots, H_N(z')) &= \big(H_{r+1}(\Phi'(z')), \dots, H_N(\Phi'(z'))\big) \\ &= \Phi''(z') = 0'' \qquad (z' \in U').\end{aligned}$$

Hence $H_j \equiv 0$, $j = r+1, \dots, N$, so that we obtain

$$\Phi''(z) = 0'' \qquad (z \in B_N). \tag{4}$$

Thus we have proved that there exist a subsequence $\{F_{n_v}\}$ (we consider $n_{v+1} - n_v$ as n_v), and an automorphism T of B_N such that $T \circ F_{n_v} \circ T^{-1}$ converge as $v \to \infty$ to a map Φ with properties (a) and (b).

Next we shall show that $\tilde{F}'(z')$ is an automorphism of B_r, where $\tilde{F} = T \circ F \circ T^{-1} = \tilde{F}' + \tilde{F}''$. We assume that $\{F_{n_v - 1}\}$ converges, without loss of generality. Putting $G = \lim_{v\to\infty} F_{n_v - 1}$ and $\tilde{G} = T \circ G \circ T^{-1} = \tilde{G}' + \tilde{G}''$ we have

$$\tilde{G} \circ \tilde{F} = \lim_{v \to \infty} T \circ F_{n_v} \circ T^{-1} = \Phi. \tag{5}$$

From this and the lemma it follows that $\tilde{G}(B_N) \subset B_N$. Hence we also obtain

$$\tilde{F} \circ \tilde{G} = \Phi. \tag{6}$$

Then we see from (1), (4), (5) and (6) that

$$\tilde{G}'\big(\tilde{F}(z')\big) = \tilde{F}'\big(\tilde{G}(z')\big) = z' \qquad (z' \in B_r) \tag{7}$$

and

$$\tilde{G}''\big(\tilde{F}(z)\big) = \tilde{F}''\big(\tilde{G}(z)\big) = 0'' \qquad (z \in B_N). \tag{8}$$

From (8) we obtain that $\tilde{F}''(z') = 0''$, $z' \in B_r$. Indeed, putting $w = \tilde{F}(z')$ we have

$$\tilde{F}''(z') = \tilde{F}''(\Phi'(z')) = \tilde{F}''(\tilde{G}(w)) = 0''.$$

Similarly we obtain that $\tilde{G}''(z') = 0''$, $z' \in B_r$. Therefore from (7) we have

$$\tilde{G}'(\tilde{F}'(z')) = \tilde{F}'(\tilde{G}'(z')) = z' \qquad (z' \in B_r).$$

This shows that the map $\tilde{F}'(z')$ is an automorphism of B_r.

If $r = N$, then F is an automorphism of B_N. Thus the proof of the theorem is complete.

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