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A NOTE ON A MAXIMAL FUNCTION OF C. FEFFERMAN AND STEIN

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ABSTRACT. We extend the class of weight functions for which a known inequality for the maximal function $T_{\lambda,r}(f)(x)$ is valid.

In this note, we derive a strengthened version of an estimate given by Torchinsky [5] and Barker [1] for the "box" maximal function of C. Fefferman and Stein [3] defined by

$$T_{\lambda,r}(f)(x) = \sup_{h>0}\left(h^{-\lambda n}\iint_{H(x,h)} t^{\lambda n-n-1}\,|f(z,t)|^r\,dz\,dt\right)^{1/r},$$

where $1<\lambda<\infty$, $0<r<\infty$, x is a point of n-dimensional Euclidean space R^n, $f(z,t)$ is a function defined in the upper half-space $R^{n+1}_+ = \{(z,t)\colon z\in R^n, 0<t<\infty\}$, and $H(x,h)$ is the region $\{(z,t)\in R^{n+1}_+\colon |x-z|<h,\ 0<t<h\}$. This maximal function has close connections with the Littlewood-Paley function g^*_λ (see [4, 5]) as well as with Peano maximal functions (see [2]).

We let $B_h(x)$ denote the ball in R^n with center x and radius h, and for $1\le d<\infty$, consider the class D_d of nonnegative measures μ on R^n which satisfy the doubling condition

$$\frac{\mu(B_h(x))}{\mu(B_s(x))}\le c\left(\frac{h}{s}\right)^{nd},\qquad 0<s\le h,\ x\in R^n.$$

We also define the Hardy-Littlewood maximal function

$$M_\mu(g)(x) = \sup_{h>0}\left(\frac{1}{\mu(B_h(x))}\int_{B_h(x)}|g(z)|\,d\mu(z)\right),$$

and the nontangential maximal function

$$N(f)(x) = \sup_{(z,t)\in\Gamma(x)}|f(z,t)|,$$

where $\Gamma(x)$ is the cone $\{(z,t)\colon |x-z|<at\}$ with vertex x and aperture a.

We will prove the following theorem.

THEOREM. *Let $\mu\in D_d$, $\lambda\ge d$, $\lambda>1$ and $r>0$. Then*

$$T_{\lambda,r}(f)(x)\le c\left\{M_\mu\left(N(f)^{rd/\lambda}\right)(x)\right\}^{\lambda/dr},$$

with c independent of f and x.

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This result was proved by Torchinsky [**5**, Theorem 1] under the additional assumption that $d\mu = w\,dx$ with $w \in A_\infty$ (the class of weight functions introduced by Muckenhoupt and C. Fefferman). We would like to point out that the result in [**5**] was also proved only in case $\lambda \geq d$. In case $w \equiv 1$, the theorem was obtained by Barker [**1**]; both [**1** and **5**] generalized an earlier result of Muckenhoupt and Wheeden [**4**].

The proof we shall give is very similar to those in [**1** and **5**], and is based on the following known lemma about Carleson measures. We use the same letter c to denote different constants.

LEMMA. *Let ν and σ be nonnegative measures on R_+^{n+1} and R^n respectively such that $\sigma(B_{2s}(\xi)) \leq c\sigma(B_s(\xi))$ for all $\xi \in R^n$ and $s > 0$. Then for $p_1 \geq p$,*

$$\left(\iint_{R_+^{n+1}} |f(z,t)|^{p_1}\, d\nu(z,t)\right)^{1/p_1} \leq c\left(\int_{R^n} N(f)(z)^p\, d\sigma(z)\right)^{1/p}$$

if and only if $\nu(H(\xi, s)) \leq c\sigma(B_s(\xi))^{p_1/p}$.

To prove the theorem, fix h and x and write

$$(1)\quad \left(h^{-\lambda n}\iint_H t^{\lambda n - n - 1}|f(z,t)|^r\,dz\,dt\right)^{1/r} = \left(\iint_{R_+^{n+1}} |\chi_H(z,t) f(z,t)|^r\, d\nu(z,t)\right)^{1/r},$$

where $H = H(x, h)$, χ_H is the characteristic function of H, and $d\nu(z,t) = h^{-\lambda n} t^{\lambda n - n - 1}\chi_H(z,t)\,dz\,dt$. Let $\sigma(\cdot) = \mu(\cdot)/\mu(B_h(x))$.

We claim that

$$\nu(H(\xi, s)) \leq c\left[\sigma(B_s(\xi))\right]^{\lambda/d}, \qquad \xi \in R^n, s > 0.$$

First note that $\nu(H(\xi, s)) = 0$ if $H(\xi, s) \cap H = \varnothing$. Also note that

$$\begin{aligned}
\nu(H(\xi,s)) &= h^{-\lambda n}\iint_{H(\xi,s)\cap H} t^{\lambda n - n - 1}\,dz\,dt \\
&= h^{-\lambda n}\left(\int_{\substack{0<t<s\\0<t<h}} t^{\lambda n - n - 1}\,dt\right)\left(\int_{\substack{|z-\xi|<s\\|z-x|<h}} dz\right) \\
&\leq c h^{-\lambda n}(\min\{s, h\})^{(\lambda n - n) + n} \quad (\text{since } \lambda > 1) \\
&= c\min\left\{\left(\frac{s}{h}\right)^{\lambda n}, 1\right\}.
\end{aligned}$$

We distinguish the cases (i) $s < h$ and (ii) $s \geq h$. In case (i), we must have $|\xi - x| < 2h$ unless $H(\xi, s) \cap H = \varnothing$. Therefore, $\mu(B_h(\xi)) \sim \mu(B_h(x))$, and by D_d we have

$$\frac{s}{h} \leq c\left[\frac{\mu(B_s(\xi))}{\mu(B_h(\xi))}\right]^{1/nd} \sim \left[\frac{\mu(B_s(\xi))}{\mu(B_h(x))}\right]^{1/nd}.$$

Hence,

$$\nu(H(\xi, s)) \leqslant c\left(\frac{s}{h}\right)^{\lambda n} \leqslant c\left[\frac{\mu(B_s(\xi))}{\mu(B_h(x))}\right]^{\lambda/d} \equiv c\sigma(B_s(\xi))^{\lambda/d}.$$

In case (ii), we must have $s \geqslant \frac{1}{2}|\xi - x|$ or else there is no intersection. Therefore, $\mu(B_s(x)) \sim \mu(B_s(\xi))$. Hence, since also $s > h$,

$$\nu(H(\xi, s)) \leqslant c \leqslant c\left[\frac{\mu(B_s(x))}{\mu(B_h(x))}\right]^{\lambda/d} \leqslant c\left[\frac{\mu(B_s(\xi))}{\mu(B_h(x))}\right]^{\lambda/d} \equiv c\sigma(B_s(\xi))^{\lambda/d}.$$

This verifies our claim.

By the lemma, since $\lambda \geqslant d$, (1) is at most

$$c\left(\int_{R^n} N(\chi_H f)(z)^{rd/\lambda}\, d\sigma(z)\right)^{\lambda/dr}.$$

Note that $\Gamma(z) \cap H = \varnothing$ unless $z \in B_{(1+a)h}(x)$. Hence,

$$N(\chi_H f)(z) \leqslant \chi_{B_{(1+a)h}(x)}(z) \cdot N(f)(z), \tag{2}$$

and (1) is bounded by

$$c\left(\frac{1}{\mu(B_h(x))}\int_{B_{(1+a)h}(x)} N(f)(z)^{rd/\lambda}\, d\mu(z)\right)^{\lambda/dr}.$$

The theorem follows by taking the supremum over h after observing that $\mu(B_h(x)) \sim \mu(B_{(1+a)h}(x))$.

Remark. If $N_h(f)(x)$ denotes the part of the nontangential maximal function with the sup restricted to the truncated cone $\Gamma(x) \cap \{(z, t)\colon 0 < t < h\}$, then (2) holds with $N(f)(z)$ on the right replaced by $N_h(f)(z)$. Hence, under the hypothesis of the theorem, the proof yields the following estimate, which is more local than that in the theorem:

$$\begin{aligned} &\left(h^{-\lambda n}\iint_{H(x,h)} t^{\lambda n - n - 1}|f(z, t)|^r\, dz\, dt\right)^{1/r} \\ &\qquad \leqslant c\left(\frac{1}{\mu(B_h(x))}\int_{B_{(1+a)h}(x)} N_h(f)(z)^{rd/\lambda}\, d\mu(z)\right)^{\lambda/dr}. \end{aligned} \tag{3}$$

We also note in passing that both this estimate and the theorem have analogues for the parabolic metrics in [5].

References

1. S. R. Barker, *An inequality for measures on a half-space*, Math. Scand. **44** (1979), 92–102.

2. S. Chanillo and R. L. Wheeden, *Inequalities for Peano maximal functions and Marcinkiewicz integrals* (to appear).

3. C. L. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

4. B. Muckenhoupt and R. L. Wheeden, *Norm inequalities for the Littlewood-Paley function* g_λ^*, Trans. Amer. Math. Soc. **191** (1974), 95–111.

5. A. Torchinsky, *Weighted norm inequalities for the Littlewood-Paley function* g_λ^*, Proc. Sympos. Pure Math., vol. 35, Amer. Math. Soc., Providence, R. I., 1979.

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