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NEW EXAMPLES OF STRICTLY ALMOST KÄHLER MANIFOLDS

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ABSTRACT. A parametrized family of non-Kähler almost Kähler manifolds is constructed as the product of solvable Lie groups with almost cosymplectic structures. A family of compact strictly almost Kähler manifolds whose cohomology is consistent with that of Kähler manifolds is similarly obtained.

Almost Kähler manifolds are almost Hermitian manifolds which carry the symplectic structure given by their closed Kähler 2-form, Φ, but which are not necessarily complex (if complex, they are Kähler). Those whose almost complex structure J is not integrable are called *strictly almost Kähler*. Few examples of strictly almost Kähler manifolds are known. In fact, only the tangent and cotangent bundles (and some related tensor bundles) of nonflat Riemannian manifolds were known to possess such structures until recently. In 1976, W. Thurston **[T]** reported a strictly almost Kähler structure on a 2-torus bundle over a 2-torus. He verified its non-Kähler status by the oddness of its first Betti number.

In **[Go]**, S. I. Goldberg studied the conjecture that the almost complex structure on a compact Einstein almost Kähler manifold is integrable. While some progress has been made, particularly upon restricting the curvature tensor **[S1, S2, S–K, Gr]**, the conjecture remains unresolved. Even in dimension four, where a normal form for the curvature of an Einstein manifold exists **[Hi, J, S–T]**, it is open. One reason may be the paucity of examples of compact strictly almost Kähler manifolds.

We announce here the construction of a large family of noncompact almost Kähler manifolds which are not Kähler and an interesting compact strictly almost Kähler 10-dimensional manifold which seeds the construction of $(6n + 4)$-dimensional spaces of like properties.

We wish to thank Allen Broughton for illuminating discussions on solvable Lie groups.

1. Almost contact metric structures. Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold **[B]**. Define the fundamental 2-form Φ in the usual way. Numerous distinct structures are definable on M in terms of Φ and η and their covariant derivatives. We are specifically interested in: (1) M is *contact* if $\Phi = d\eta$; (2) M is *normal* if $[\phi, \phi] + 2d\eta \otimes \xi = 0$; (3) M is *Sasakian* if it is normal and contact; (4) M is *almost cosymplectic* if $d\Phi = 0$ and $d\eta = 0$; (5) M is *cosymplectic* if $\nabla\phi = 0$.

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Obviously, an almost cosymplectic manifold cannot be contact. A normal almost cosymplectic manifold is cosymplectic.

2. Calabi-Eckmann-Morimoto products. In **[C–E]**, E. Calabi and B. Eckmann announced an integrable complex structure on the Cartesian product of two odd-dimensional spheres, $S^{2p+1} \times S^{2q+1}$, generalizing the construction of H. Hopf on $S^1 \times S^{2n+1}$ **[Ho]**. Since the second Betti number of these compact manifolds is zero, they cannot carry a Kähler structure. About a decade later, A. Morimoto **[M]** generalized their construction to encompass the product of two almost contact metric manifolds. Although Morimoto did not use the a, b-parametrization of J and g which Calabi and Eckmann used, only minor changes are necessary in Morimoto's results upon assuming the greater generality. Morimoto proved that the induced almost complex structure on the product manifold is integrable if and only if *each* of the two almost contact metric structures is normal (Proposition 1 below). Thus, the Morimoto product of the two Sasakian spheres, S^{2p+1} and S^{2q+1}, $p \geqslant 1$, $q \geqslant 1$, is Hermitian as Calabi and Eckmann found.

To be specific, let $(M_1^{2m_1+1}, \phi_1, \xi_1, \eta_1, g_1)$ and $(M_2^{2m_2+1}, \phi_2, \xi_2, \eta_2, g_2)$ be almost contact metric manifolds. Then for real a and nonzero real b, define $M_{a,b}^{m_1,m_2} = (M_1 \times M_2, J, g)$ via

$$J(X_1, X_2) = \big(\phi_1 X_1 - \{(ab^{-1})\eta_1(X_1) + (a^2 + b^2)(b^{-1})\eta_2(X_2)\}\xi_1,\\ \phi_2 X_2 + \{(b^{-1})\eta_1(X_1) + (ab^{-1})\eta_2(X_2)\}\xi_2\big)$$

and

$$g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + a\eta_1(X_1)\eta_2(Y_2) + a\eta_1(Y_1)\eta_2(X_2)\\ + (a^2 + b^2 - 1)\eta_2(X_2)\eta_2(Y_2) + g_2(X_2, Y_2).$$

It is an easy exercise to verify that g is Hermitian with respect to J. Extending Φ_i and η_i to the product, we calculate

$$\Phi = \Phi_J = \Phi_1 + \Phi_2 + 2b\eta_1 \wedge \eta_2.$$

PROPOSITION 1 **[M]**. *J is integrable if and only if both ϕ_1 and ϕ_2 are normal.*

PROPOSITION 2. $d\Phi = d\Phi_1 + d\Phi_2 + 2b\{d\eta_1 \wedge \eta_2 - \eta_1 \wedge d\eta_2\}$.

Propositions 1 and 2 imply that the Morimoto product of two strictly almost cosymplectic manifolds is strictly almost Kähler.

3. Examples. Let $\bar{a} = (a_1, \ldots, a_{2n})$ be a $2n$-tuple of real numbers, $\bar{a} \neq (0, \ldots, 0)$, and let V^{2n+1} be a real vector space with chosen basis $\{e_0, e_1, \ldots, e_{2n}\}$. Let $\mathfrak{G}_{\bar{a}}$ denote the Lie algebra constructed on V via:

(i) $[e_i, e_j] = 0$, $i \neq 0, j \neq 0$,

(ii) $[e_0, e_i] = a_i e_i + a_{i+n} e_{i+n}$, $i = 1, \ldots, n$,

(iii) $[e_0, e_{i+n}] = a_{i+n} e_i - a_i e_{i+n}$, $i = 1, \ldots, n$.

$\mathfrak{G}_{\bar{a}}$ is a solvable Lie algebra isomorphic to the semidirect product of its $2n$-dimensional maximal abelian ideal and the subalgebra generated by e_0. Let $G_{\bar{a}}$ be a connected real Lie group whose Lie algebra is $\mathfrak{G}_{\bar{a}}$. Olszak [**O**] defined a noncosymplectic almost cosymplectic structure on $G_{\bar{a}}$ as follows. The $(2n+1)$-dimensional identity matrix supplies a left invariant Riemannian metric for $G_{\bar{a}}$ and we take its Levi-Civita connection. Set $\eta(e_i) = \delta_{0i}$, for $i = 0, 1, \ldots, 2n$. Then we may take $\xi = e_0$. Define $\varphi(e_0) = 0$, $\varphi(e_i) = e_{i+n}$ and $\varphi(e_{i+n}) = -e_i$, for $i = 1, \ldots, n$. Then $(G_{\bar{a}}, \varphi, \xi, \eta, g)$ is an almost cosymplectic metric manifold. Since $\nabla\varphi \neq 0$, $G_{\bar{a}}$ is not cosymplectic. Thus we have

THEOREM 3. *Given two odd integers, $2p+1$ and $2q+1$, p, $q > 0$, there exist noncompact, connected, solvable Lie groups, G_1^{2p+1} and G_2^{2q+1}, such that $G^{p,q} \cong G_1 \times G_2$ carries a two parameter family of almost Kähler structures which are not Kähler.*

As a second example, let W^4 denote Thurston's torus bundle over T^2 with strictly almost Kähler structure (J, g). Give $W \times S^1$ the almost cosymplectic structure [**G–Y**]: $\phi(X, Y) = (JX, 0)$, $\xi = (0, d/dt)$ and $\eta = (0, dt)$. $W \times S^1$ is not cosymplectic because W is not Kähler. Construct on $M = (W \times S^1) \times (W \times S^1)$ the non-Kähler almost Kähler structure described herein. We note that we could not have concluded the non-Kähler property of $M_{a,b}$ from its cohomology because:

$b_1(M) = 8$,
$b_2(M) = 22 + 2b_2(W) = 30$,
$b_3(M) = 30 + 10b_2(W) = 70$,
$b_4(M) = 41 + 14b_2(W) + b_2(W)^2 = 113$,
$b_5(M) = 52 + 12b_2(W) + 2b_2(W)^2 = 132$.

Clearly, $(M \times S^1) \times (W \times S^1)$, etc.

If N is any compact Kähler manifold, $M = (W \times S^1) \times (N \times S^1)$ can be made strictly almost Kähler in the same way. However, in this case, $b_1(M) = b_1(N) + 5$ is odd and the non-Kähler property emerges from cohomological considerations. Similarly for $(M \times S^1) \times (W \times S^1)$, etc.

REFERENCES

[**B**] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., vol. 509, Springer-Verlag, New York, 1971.

[**C–E**] E. Calabi and B. Eckmann, *A class of compact complex manifolds which are not algebraic*, Ann. of Math. (2) **58** (1953), 494–500.

[**Go**] S. I. Goldberg, *Integrability of almost Kähler manifolds*, Proc. Amer. Math. Soc. **21** (1969), 96–100.

[**G–Y**] S. I. Goldberg and K. Yano, *Integrability of almost cosymplectic structures*, Pacific J. Math. **31** (1969), 373–382.

[**Gr**] A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. **28** (1976), 601–612.

[**Hi**] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geom. **9** (1974), 435–441.

[**Ho**] H. Hopf, *Zur topologie der komplexen mannigfaltigkeiten*, Studies and Essays Presented to R. Courant, Interscience, New York, 1948, pp. 167–185.

[**J**] D. L. Johnson, *A normal form for curvature*, Thesis, M.I.T., 1977.

[**M**] A. Morimoto, *On normal almost complex structures*, J. Math. Soc. Japan **15** (1963), 420–436.

[**O**] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math. J. **1** (1981), 239–250.

[S1] S. Sawaki, *On almost Hermitian manifolds satisfying a certain condition on the almost complex structure tensor*, Differential Geometry (in Honor of Kentaro Yano), Kinokuniya, Tokyo, 1972, pp. 443–450.

[S2] ______, *Sufficient conditions for an almost Hermitian manifold to be Kählerian*, Hokkaido Math. J. **1** (1972), 21–29.

[S–K] S. Sawaki and S. Kotô, *Almost complex metric spaces*, Sûgaku **16** (1964), 9–18.

[S–T] I. M. Singer and J. Thorpe, *The curvature of 4-dimensional Einstein spaces*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 355–365.

[T] W. P. Thurston, *Some examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467–468.

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