PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 3, July 1983

NO HOMOGENEOUS TREE-LIKE CONTINUUM CONTAINS AN ARC

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ABSTRACT. In 1960 R. H. Bing [**4**, p. 210] asked the following question. "Is there a homogeneous tree-like continuum that contains an arc?" We answer this question in the negative. This result generalizes Bing's theorem [**4**, p. 229] that every atriodic homogeneous tree-like continuum is arcless.

1. Introduction. Is every homogeneous tree-like continuum a pseudo-arc? This classic unsolved problem is one of the principal obstacles to the classification of homogeneous 1-dimensional continua [**8**].

Suppose there exists a homogeneous tree-like continuum M that is not a psuedo-arc. F. B. Jones [**14**] proved that M is indecomposable. Bing [**3**] proved that M is not arc-like. C. E. Burgess [**6**] proved that M is not circle-like. It follows from results of Burgess [**7**] and I. W. Lewis [**15**] that M is not k-junctioned and that M has a proper subcontinuum that is not a pseudo-arc. In this paper we prove that M does not contain an arc.

2. Definitions and related results. A space is *homogeneous* if for each pair x, y of its points there is a homeomorphism of the space onto itself that takes x to y.

A *chain* is a finite collection $\{L_i\colon 1 \leqslant i \leqslant n\}$ of open sets such that $L_i \cap L_j \neq \varnothing$ if and only if $|i-j| \leqslant 1$. If $n > 2$ and L_1 also intersects L_n, the collection is called a *circular chain*. Each L_i is called a *link*.

A collection $\mathfrak{R}$ of sets is *coherent* if for each nonempty proper subcollection $\mathfrak{S}$ of $\mathfrak{R}$, an element of $\mathfrak{S}$ intersects an element of $\mathfrak{R}\setminus\mathfrak{S}$.

Following Bing [**2**, p. 653], we define a finite coherent collection $\mathfrak{T}$ of open sets to be a *tree chain* if no three elements of $\mathfrak{T}$ have a point in common and no subcollection of $\mathfrak{T}$ is a circular chain.

A *continuum* is a nondegenerate compact connected metric space.

A continuum M is *arc-like* (*circle-like*, *tree-like*, respectively) if for each positive number ε there is a chain (circular chain, tree chain) covering M such that each element has diameter less than ε.

A continuum is *decomposable* if it is the union of two of its proper subcontinua; otherwise, it is *indecomposable*. If a continuum does not have a decomposable

Received by the editors October 19, 1982 and, in revised form, December 3, 1982.

1980 *Mathematics Subject Classification.* Primary 54F15, 54F20; Secondary 54F50, 54F55.

Key words and phrases. Homogeneity, tree-like continuum, arcless continuum, tree chain, hereditarily indecomposable continuum, pseudo-arc, dog-chases-rabbit principle.

[1] The author was partially supported by NSF Grant MCS 8205282.

subcontinuum it is *hereditarily indecomposable*. A continuum M is a *triod* if M has a subcontinuum H such that $M \setminus H$ is the union of three nonempty disjoint open sets. If a continuum does not contain a triod it is *atriodic*. Since the plane does not contain uncountably many disjoint triods [**17**], every indecomposable homogeneous plane continuum is atriodic [**10**, Lemma 1].

Every homogeneous tree-like plane continuum is hereditarily indecomposable [**11**]. In fact, no homogeneous tree-like atriodic continuum has a decomposable subcontinuum [**13**, Corollary 2]. Furthermore, every homogeneous hereditarily indecomposable continuum is tree-like [**18**]. It would be interesting to know if every homogeneous tree-like continuum is hereditarily indecomposable [**4**, p. 210].

3. Preliminaries. Henceforth M is a homogeneous continuum with metric ρ.

Let ε be a positive number. A homeomorphism h of M onto M is called an *ε-homeomorphism* if $\rho(x, h(x)) < \varepsilon$ for each point x of M.

Notation. Let x be a point of M. We denote the set $\{y \in M$: an ε-homeomorphism of M onto M takes x to $y\}$ by $W(x, \varepsilon)$.

LEMMA. *For every positive number ε and every point x of M, the set $W(x, \varepsilon)$ is open in M.*

PROOF. This lemma follows from a short argument [**10**, Lemma 4, proof] involving E. G. Effros' topological transformation group theorem [**9**, Theorem 2.1].

Notation. Let x be a point of M. Let r and s be numbers such that $0 < r < s$. We denote the set $\{y \in M: r \leqslant \rho(x, y) \leqslant s\}$ by $A(x, r, s)$. The set $\{y \in M: \rho(x, y) \leqslant r\}$ is denoted by $B(x, r)$. We denote $\{y \in M: \rho(x, y) = r\}$ by $C(x, r)$.

4. Our result. In [**4**, p. 221] Bing proved that every homogeneous tree-like continuum has a proper subcontinuum that is not an arc. He then used this theorem to prove [**4**, p. 229] that no homogeneous tree-like atriodic continuum contains an arc. He also asked, "Is there a homogeneous tree-like continuum that contains an arc?" [**4**, p. 210]. We use the dog-chases-rabbit principle [**5**, p. 123] to answer Bing's question in the negative.

THEOREM. *If M is a homogeneous tree-like continuum, then M does not contain an arc.*

PROOF. Assume M contains an arc. Let $r_1, r_2, \ldots$ be a strictly decreasing sequence of numbers that converges to 0. For each positive integer i, let $X_i = \{x \in M$: x belongs to an arc in M that has both endpoints in $C(x, r_i)\}$.

It follows from the homogeneity of M that

(1) $$M = \bigcup \{X_i: i = 1, 2, \ldots\}.$$

For each i,

(2) X_i is in the interior of X_{i+1} relative to M.

To see this let x be a point of X_i, let $\varepsilon = (r_i - r_{i+1})/2$, and note that the open set $W(x, \varepsilon)$ of our Lemma is in X_{i+1}.

It follows from (1), (2), and the compactness of M that $X_j = M$ for some positive integer j. Assume without loss of generality that

$$(3) \qquad r_j > 10.$$

For each i, let Y_i be the set of all points x of M with the following property. There exist arcs J and K in $B(x, 9 + 1/i)$ from x to $C(x, 9 + 1/i)$ such that

$$\rho(J \cap A(x, 2 - 1/i, 9 + 1/i), K \cap A(x, 2 - 1/i, 9 + 1/i)) > 1/i.$$

It follows from (3) that

$$(4) \qquad M = \bigcup \{Y_i : i = 1, 2, \dots\}.$$

For each i,

(5) $\qquad Y_i$ is in the interior of Y_{i+1} relative to M.

To see this let x be a point of Y_i. Let $\varepsilon = (1/i - 1/(i+1))/2$. For each point y of $B(x, \varepsilon)$,

$$\rho(A(y, 2 - 1/(i+1), 9 + 1/(i+1)),\ M \setminus A(x, 2 - 1/i, 9 + 1/i)) \geqslant \varepsilon.$$

It follows that the open set $W(x, \varepsilon)$ is in Y_{i+1}. Hence (5) is true.

It follows from (4), (5) and the compactness of M that for some positive integer k,

$$(6) \qquad Y_k = M.$$

Let $\mathfrak{T}$ be a tree chain covering M such that each element of $\mathfrak{T}$ has diameter less than $1/k$. Let x_0 be a point of M. By (3), $C(x_0, 9) \neq \varnothing$. Let $\mathcal{C}_0$ be a chain in $\mathfrak{T}$ that goes from x_0 to a point x_1 of $C(x_0, 9)$.

By (6), there exist arcs J and K in $B(x_1, 9)$ from x_1 to $C(x_1, 9)$ such that

$$(7) \qquad \rho(J \cap A(x_1, 2, 9), K \cap A(x_1, 2, 9)) > 1/k.$$

Note that

(8) $\qquad$ either J or K misses $A(x_1, 3, 9) \cap \bigcup \mathcal{C}_0$.

To see this assume the contrary. Let $\mathfrak{J}$ be a subchain of $\mathcal{C}_0$ that goes from a point of $J \cap A(x_1, 3, 9)$ to x_1. Let $\mathcal{K}$ be a subchain of $\mathcal{C}_0$ that goes from a point of $K \cap A(x_1, 3, 9)$ to x_1. Either $\mathfrak{J}$ is a subchain of $\mathcal{K}$ or $\mathcal{K}$ is a subchain of $\mathfrak{J}$. Assume without loss of generality that $\mathfrak{J}$ is a subchain of $\mathcal{K}$. Since $\mathfrak{T}$ is a tree chain, K intersects each link of $\mathcal{K}$. Hence the first link of $\mathfrak{J}$ intersects both $J \cap A(x_1, 2, 9)$ and $K \cap A(x_1, 2, 9)$, and this contradicts (7). Thus (8) is true.

Assume without loss of generality that

$$(9) \qquad J \cap A(x_1, 3, 9) \cap \bigcup \mathcal{C}_0 = \varnothing.$$

Let x_2 be the endpoint of J that belongs to $C(x_1, 9)$. Let $\mathcal{C}_1$ be the chain in $\mathfrak{T}$ that goes from x_1 to x_2. Since $\mathfrak{T}$ is a tree chain,

(10) $\qquad$ each link of $\mathcal{C}_1$ intersect J.

Let $\mathfrak{D}_1$ be the collection of links of $\mathcal{C}_1$ that intersect $A(x_1, 4, 9)$.

The sets $\bigcup \mathfrak{D}_1$ and $\bigcup \mathcal{C}_0$ are disjoint; for otherwise, by (9) and (10), there exists a circular chain in $\mathcal{C}_0 \cup \mathcal{C}_1$, and this contradicts the fact that $\mathfrak{T}$ is a tree chain. Hence $\mathfrak{D}_1 \setminus \mathcal{C}_0 \neq \varnothing$.

We proceed inductively. Let n be a positive integer. Assume that for each positive integer m less than or equal to n, we have defined points x_m and x_{m+1} of M and subcollections $\mathcal{C}_m$ and $\mathcal{D}_m$ of $\mathcal{T}$ such that

(11_{m}) $$x_{m+1} \in C(x_m, 9),$$

(12_{m}) $$\mathcal{C}_m \text{ is a chain from } x_m \text{ to } x_{m+1},$$

(13_{m}) $$\mathcal{D}_m \text{ is the collection of links of } \mathcal{C}_m \text{ that intersect } A(x_m, 4, 9),$$

(14_{m}) $$\left(\bigcup \mathcal{D}_m\right) \cap \left(\bigcup \bigcup \{\mathcal{C}_i : 0 \leq i < m\}\right) = \varnothing, \text{ and}$$

(15_{m}) $$\mathcal{D}_m \setminus \bigcup \{\mathcal{C}_i : 0 \leq i < m\} \neq \varnothing.$$

By (6), there exist arcs J' and K' in $B(x_{n+1}, 9)$ from x_{n+1} to $C(x_{n+1}, 9)$ such that

(16) $$\rho(J' \cap A(x_{n+1}, 2, 9), K' \cap A(x_{n+1}, 2, 9)) > 1/k.$$

It follows from (16) and the argument for (8) that either J' or K' misses $A(x_{n+1}, 3, 9) \cap \bigcup \mathcal{C}_n$. Assume without loss of generality that

(17) $$J' \cap A(x_{n+1}, 3, 9) \cap \bigcup \mathcal{C}_n = \varnothing.$$

Let x_{n+2} be the endpoint of J' that belongs to $C(x_{n+1}, 9)$. Let $\mathcal{C}_{n+1}$ be the chain in $\mathcal{T}$ from x_{n+1} to x_{n+2}. Since $\mathcal{T}$ is a tree chain,

(18) $$\text{each link of } \mathcal{C}_{n+1} \text{ intersects } J'.$$

Let $\mathcal{D}_{n+1}$ be the collection of links of $\mathcal{C}_{n+1}$ that intersect $A(x_{n+1}, 4, 9)$.
Since $\mathcal{T}$ is a tree chain, it follows from (17) and (18) that

(19) $$\left(\bigcup \mathcal{D}_{n+1}\right) \cap \left(\bigcup \mathcal{C}_n\right) = \varnothing.$$

Furthermore,

(20) $$\left(\bigcup \mathcal{D}_{n+1}\right) \cap \left(\bigcup \bigcup \{\mathcal{C}_i : 0 \leq i < n\}\right) = \varnothing.$$

To see this suppose an element L of $\mathcal{D}_{n+1}$ intersects $\bigcup \bigcup \{\mathcal{C}_i : 0 \leq i < n\}$. Let $\mathcal{E}$ be a chain in $\bigcup \{\mathcal{C}_i : 0 \leq i < n\}$ such that L intersects one end link of $\mathcal{E}$ and x_n belongs to the other end link of $\mathcal{E}$. Let $\mathcal{F}$ be the subchain of $\mathcal{C}_{n+1}$ with the property that x_{n+1} belongs to one end link of $\mathcal{F}$ and the other end link of $\mathcal{F}$ is L.

By (14_{n}), links of $\mathcal{E}$ intersect links of $\mathcal{C}_n$ only in $B(x_n, 4)$. By (19), links of $\mathcal{F}$ intersect links of $\mathcal{C}_n$ only in $B(x_{n+1}, 4)$. Therefore, since $B(x_n, 4)$ and $B(x_{n+1}, 4)$ are disjoint, there exists a circular chain in $\mathcal{C}_n \cup \mathcal{E} \cup \mathcal{F}$. This contradicts the fact that $\mathcal{T}$ is a tree chain. Hence (20) is true.

By (19) and (20), $\mathcal{D}_{n+1} \setminus \bigcup \{\mathcal{C}_i : 0 \leq i \leq n\} \neq \varnothing$. This completes the inductive step.

For each positive integer m, there exist points x_m and x_{m+1} of M and subcollections $\mathcal{C}_m$ and $\mathcal{D}_m$ of $\mathcal{T}$ that satisfy (11_{m})–(15_{m}). Consequently $\mathcal{T}$ is infinite, and this contradicts the fact that $\mathcal{T}$ is a tree chain. Therefore M does not contain an arc.

Comments. This argument can be modified to prove that no homogeneous tree-like continuum contains two continua with only one point in common. However the question of whether every homogeneous tree-like continuum is hereditarily indecomposable remains open.

A continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs [**12**]. No tree-like continuum contains a solenoid. Perhaps every homogeneous continuum that contains an arc contains a solenoid.

ACKNOWLEDGMENT. The author wishes to thank Mark Marsh, Jim Rogers, and Eldon Vought for several interesting conversations about homogeneous continua.

ADDENDUM. Recently Lewis [**16**] proved that if there is a homogeneous hereditarily indecomposable tree-like continuum that is not a pseudo-arc, then there is one with no arc-like subcontinua.

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