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ON THE TRANSPOSE MAP OF MATRIX ALGEBRAS

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ABSTRACT. It is shown that for the transpose map $\theta(n)$ of the $n \times n$ matrix algebra M_n, its kth multiplicity map $\theta(n)_k$ has exactly the norm k if $k \leqslant n$, hence the completely bounded norm of $\theta(n)$ written $\|\theta(n)\|_{cb}$ equals n. Some applications and related results are also proved.

Let M_n be the $n \times n$ matrix algebra over the complex field. By M_∞ we mean the algebra of all bounded linear operators on an infinite-dimensional Hilbert space expressed as the algebra of matrices of infinite order. Let $\theta(n)$ $(n = 1, 2, 3, \ldots)$ be the transpose map in M_n. The map $\theta(n)$ is an anti-*-automorphism of M_n. In this note we shall show some further properties of $\theta(n)$, which may contribute to the theory of completely bounded maps between C^*-algebras.

1. Throughout this paper we consider M_n as the C^*-algebra of all bounded linear operators on an n-dimensional Hilbert space. The tensor product $M_n \otimes M_k$ is then regarded not only as the matrix C^*-algebra $M_k(M_n)$ over the algebra M_n but also as the matrix C^*-algebra $M_n(M_k)$ over M_k. The kth multiplicity map $\theta(n)_k$ on $M_n \otimes M_k = M_k(M_n)$ is defined by

$$\theta(n)_k[a_{ij}] = \left[\theta(n)(a_{ij})\right] \quad \text{on } M_k(M_n).$$

This is also regarded as the transpose map $[b_{ij}] \to [b_{ji}]$ on the algebra $M_n(M_k)$. The following lemma is more or less known. We omit the proof.

LEMMA 1.1. *Let $M_n(A)$ be the $n \times n$ matrix algebra over a C^*-algebra A and $[a_{ij}]$ be an element of $M_n(A)$. Then, we have that*

$$\|[a_{ij}]\| \leqslant \left(\sum_{i,j=1}^{n} \|a_{ij}\|^2 \right)^{1/2}.$$

We note that the above estimate is attained in some cases. In fact, the norm of the matrix $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ is equal to 2 and equality also can occur for examples of matrices of any order n.

THEOREM 1.2. *The norms of the multiplicity maps $\theta(n)$ $(n = 1, 2, 3, \ldots, \infty)$ are*

$$\|\theta(n)_k\| = \begin{cases} k, & \text{if } k \leqslant n, \\ n, & \text{if } k > n. \end{cases}$$

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PROOF. Let $[a_{ij}]$ be an element of the algebras $M_k(M_n)$ with $\|[a_{ij}]\| \leqslant 1$. Then $\|a_{ij}\| \leqslant 1$ for every i and j. Since $\theta(n)$ is an isometry, we have that $\|\theta(n)(a_{ij})\| \leqslant 1$ for every i and j and hence, by Lemma 1.1,

$$\|\theta(n)_k([a_{ij}])\| = \|[\theta(n)(a_{ij})]\| \leqslant k.$$

Next we consider the map $\theta(n)_k$ as the transpose map of the algebra $M_n(M_k) = M_k(M_n)$ and assume that $k \leqslant n$ (for a finite n). Let $\{e_{ij}\}$ be the matrix units of M_k. It is then not so difficult to see that the norm of the matrix

$$x = \left[\begin{array}{cccc|c} e_{11}, & e_{21}, & \dots, & e_{k1}, & \\ e_{12}, & e_{22}, & \dots, & e_{k2}, & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ e_{1k}, & e_{2k}, & \dots, & e_{kk}, & \\ \hline & & 0 & & 0 \end{array}\right]$$

is one, whereas the norm of the matrix $\theta(n)_k(x)$ is k because x is a selfadjoint partial isometry and $\theta(n)_k(x)/k$ is a projection of the algebra $M_n(M_k)$. Therefore, $\|\theta(n)_k\| \geqslant k$ and $\|\theta(n)_k\| = k$.

On the other hand, for every k, one sees that $\|\theta(n)_k\| \leqslant n$ by Lemma 1.1 in $M_n(M_k)$. Hence, $\|\theta(n)_k\| = n$ for every $k > n$. This completes the proof.

The estimate $\|\theta(\infty)_k\| \geqslant \sqrt{k}$ for every k was pointed out by Okayasu in [**7**] and used to show that the minimal C^*-crossnorm is not a uniform crossnorm.

A linear map τ between C^*-algebras A and B is said to be completely bounded if the norms of its multiplicity maps $\{\tau_k \mid k = 1, 2, 3, \dots\}$ are bounded. It has been recently recognized in the literature ([**2, 3, 9**], etc.) that the appropriate linear maps to attach the matricial structure of C^*-algebras are not merely bounded maps but completely bounded maps. We write $\|\tau\|_{cb} = \sup\|\tau_k\|$ for a completely bounded map τ. In this terminology, the map $\theta(\infty)$ is precisely an example of a bounded linear map on a C^*-algebra which is not completely bounded. Moreover, the norms of the multiplicity maps $\theta(\infty)_k$'s are strictly increasing, with $\|\theta(\infty)_k\| = k$. As an application of the above theorem we can give an answer to a question in Loebl [**6**]. Namely, we have the following.

THEOREM 1.3. *Let B be a C^*-algebra. Then the following assertions are equivalent*:

(1) *For any C^*-algebra A, every bounded map from A to B is completely bounded.*

(2) *B is a C^*-subalgebra of a matrix C^*-algebra $M_n(C)$ for some commutative C^*-algebra C and for some n.*

PROOF. The implication (2) $\Rightarrow$ (1) is shown in [**6**, Lemma 7]. For the implication (1) $\Rightarrow$ (2) we assume that B is acting on the atomic representation space; that is, the weak closure $\tilde{B}$ of B is the direct sum of the algebras of all bounded operators on Hilbert spaces, $\tilde{B} = \Sigma_\alpha \oplus L(H_\alpha)$. Let θ^α be the transpose map of $L(H_\alpha)$ and consider the map $\theta = \Sigma_\alpha \oplus \theta^\alpha$ on $\tilde{B}$. Put $A = \theta(B)$; then as $\theta^2 =$ identity, the map $\tau = \theta|_A$ is a bounded linear map from A to B. Now suppose that B does not satisfy condition (2). Then it has irreducible representations with arbitrarily big dimensions.

Since

$$\|\tau_k\| = \|\theta_k\| \geqslant \|(\theta^\alpha)_k\| \quad \text{for every } \theta^\alpha,$$

we see that τ is not completely bounded by Theorem 1.2.

The author and T. Huruya [**4**] have recently determined necessary and sufficient conditions on a pair of C^*-algebras that ensure that every bounded linear map between them is also completely bounded.

It is known that every unital completely contractive map is necessarily completely positive [**1**, Proposition 1.2.8], but one should notice that the result is not valid for nonunital maps. In fact, the map $\tau = \theta(n)/n$ in M_n is completely contractive and in fact positive but it is not even 2-positive.

2. In this section we study properties of the (generalized) transpose map $\theta(n, A)$ of the algebra $M_n(A)$ over a C^*-algebra A. We first note that $\theta(n, A)$ is merely the restriction of the multiplicity map $\theta(n)_k$ for some k (possibly $k = \infty$). In this connection we recall the following fact which is more or less known and we leave its proof to the readers.

PROPOSITION 2.1. *Let τ be a linear map of a C^*-algebra A to a C^*-algebra B. The map τ is then completely bounded if and only if for every C^*-algebra C there exists a unique bounded product map $\tau \otimes 1$ from the C^*-tensor product $A \otimes_{\min} C$ to $B \otimes_{\min} C$ such that $\tau \otimes 1(a \otimes b) = \tau(a) \otimes b$ for every $a \in A$ and $b \in C$. Moreover, if C has irreducible representations of arbitrarily large dimensions, then $\|\tau \otimes 1\| = \|\tau\|_{cb} = \|\tau \otimes 1\|_{cb}$.*

Along the same lines as Theorem 1.2 one may easily verify that $\|\theta(n, A)\|_{cb} = \|\theta(n)\|_{cb} = n$, and that $\|\theta(n, A)\| = n$ or k according to whether A has an irreducible repesentation with $\dim \pi \geqslant n$ or whether it has only at most k-dimensional irreducible representations.

The map $\theta(2)$ is often used as an example of a positive map which is not completely positive (specifically, not 2-positive). This situation is clarified by the following.

THEOREM 2.2. *The transpose map $\theta(n, A)$ on $M_n(A)$ $(n \geqslant 2)$ is positive if and only if A is commutative.*

PROOF. Let $a = [a_{ij}]$ be a positive matrix in $M_n(A)$. To show that $\theta(n, A)$ is a positive map, we may assume (cf. [**5**, Proposition 2.1]) that $a_{ij} = a_i^* a_j$ for n elements $\{a_1, a_2, \ldots, a_n\}$ in A. If A is commutative the (i, j)-entry of $\theta(n, A)(a)$ is written as $a_j^* a_i = (a_i^*)^* a_j^*$, whence the matrix $[a_j^* a_i]$ is positive.

Conversely suppose that A is not commutative. We consider the double transpose of $\theta(n, A)$ which becomes the transpose map $\theta(n, \tilde{A})$ in the matrix algebra $M_n(\tilde{A})$ over the universal enveloping von Neumann algebra $\tilde{A}$ of A. It suffices then to show that $\theta(n, \tilde{A})$ is not positive. By the assumption, there exist orthogonal equivalent projections p and q in $\tilde{A}$ with a partial isometry u implementing their equivalence.

Let x be a positive matrix of $M_n(\tilde{A})$ defined as

$$x = \left[\begin{array}{cc|c} p & u^* & \\ u & q & 0 \\ \hline & 0 & 0 \end{array}\right]$$

Then one can verify that the matrix $\theta(n, \tilde{A})(x)$ is not positive (cf. [**8**, Proof of Proposition 3.1]). This completes the proof.

Thus in particular the usual transpose map $\theta(n)$ is a positive map that is not 2-positive for $n \geqslant 2$.

The theorem implies the following more general result.

COROLLARY 2.3. *Let A be a commutative C^*-algebra, then the transpose map $\theta(n, A)$ $(n \geqslant 2)$ on $M_n(A)$ is a positive map which is not* 2*-positive.*

REFERENCES

1. W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224.
2. E. Christensen, *Extensions of derivations*. II, Math. Scand. **50** (1982), 111–122.
3. U. Haagerup, *Solution of similarity problem for cyclic representations of C^*-algebras*, preprint.
4. T. Huruya and J. Tomiyama, *Completely bounded maps of C^*-algebras*, J. Operator Theory (to appear).
5. C. Lance, *On nuclear C^*-algebras*, J. Funct. Anal. **12** (1973), 157–176.
6. R. T. Loebl, *Contractive linear maps on C^*-algebras*, Michigan Math. J. **22** (1975), 361–366.
7. T. Okayasu, *Some crossnorms which are not uniform cross*, Proc. Japan Acad. Ser. A Math. **46** (1970), 54–57.
8. T. Takasaki and J. Tomiyama, *Stinespring type theorems for various types of completely positive maps associated to operator algebras*, Math. Japon. **271** (1982), 129–139.
9. G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. **40** (1981), 127–150.

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