PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

DISCRETE SETS OF SINGULAR CARDINALITY

WILLIAM G. FLEISSNER

ABSTRACT. Let κ be a singular cardinal. In Fleissner's thesis, he showed that in normal spaces X, certain discrete sets Y of cardinality κ (called here sparse) which are $< \kappa$-separated are, in fact, separated. In Watson's thesis, he proves the same for countably paracompact spaces X. Here we improve these results by making no assumption on the space X. As a corollary, we get that assuming $V = L$, $\aleph_1$-paralindelöf T_2 spaces of character $\leqslant \omega_2$ are collectionwise Hausdorff.

In his thesis, Fleissner proved

THEOREM [**F**]. *Assuming $V = L$, normal, T_2, spaces of character $\leqslant \mathbf{c}$ are collectionwise Hausdorff.*

The proof is by induction on κ, the induction hypothesis being that discrete sets of cardinality κ can be separated. For κ regular, the proof uses a $\diamondsuit$-like principle. For κ singular, the induction hypothesis GCH and normality are used to show that a discrete set of cardinality κ is sparse (defined below). The singular κ case is finished by proving that in normal spaces X, discrete, sparse, $< \kappa$-separated sets are separated. (Let us call this the last lemma.) In his thesis [**W**], Watson proved the analogous results with normality replaced with countable paracompactness. Here we prove the last lemma without assuming that X is either normal or countably paracompact.

A subset Y of a space $(X, \mathfrak{T})$ is called *discrete* if every $x \in X$ has a neighborhood containing at most one point of Y. A *neighborhood assignment* for Y is a function U: $Y \to \mathfrak{T}$ such that for all $y \in Y$, $y \in U(y)$. Y is *separated* if there is a disjoint neighborhood assignment for Y. Y is $<\kappa$*-separated* if every subset of Y of cardinality $< \kappa$ is separated.

Let us fix a singular cardinal κ and a closed, cofinal in κ, set of cardinals, $\{\kappa_\beta$: $\beta < \mathrm{cf}(\kappa)\}$, enumerated in increasing order, such that $\kappa_0 = 0$ and $\kappa_1 \geqslant \mathrm{cf}\,\kappa$, $\kappa_1 > \omega_1$. Throughout this paper, Y will be a discrete subset of a space X with $|Y| = \kappa$. We say that $\mathcal{Q} = (A_\beta)_{\beta<\kappa}$ is a *nice chain* if $\bigcup \mathcal{Q} = Y$; for all $\beta < \kappa$, $|A_\beta| = \kappa_\beta$; if $\alpha < \beta$, then $A_\alpha \subset A_\beta$; and for limit ordinals λ, $\bigcup \{A_\beta\colon \beta < \lambda\} = A_\lambda$.

Given a nice chain $\mathcal{Q}$, a neighborhood assignment U, and a $\beta < \mathrm{cf}\,\kappa$, we define

$$S(\mathcal{Q}, U, \beta) = \overline{\bigcup \{U(y)\colon y \in A_\beta\}} \cap (Y - A_\beta).$$

Received by the editors September 1, 1982 and, in revised form, November 22, 1982.
1980 *Mathematics Subject Classification*. Primary 54D15, 03E05.
Key words and phrases. Discrete, singular cardinals, collectionwise Hausdorff.

We will say that U *is thin w.r.t.* $\mathcal{C}$ if, for all $\beta < \mathrm{cf}\,\kappa$, $|\,S(\mathcal{C}, U, \beta)\,| \leq \kappa_\beta$. We will say that Y is *sparse* if for every nice chain $\mathcal{C}$ there is a neighborhood assignment U which is thin w.r.t. $\mathcal{C}$.

The notion "sparse" is rather technical, but it is an important intermediate concept, as illustrated by the following two lemmata.

LEMMA 1. *Assume GCH. Let Y be a discrete subset with singular cardinality κ of a space X with the character of X less than κ. If X is* (a) *normal, or* (b) *countably paracompact, or* (c) *$\aleph_1$-paralindelöf, then Y is sparse.*

LEMMA 2. *If Y is sparse and $< \kappa$-separated, then Y is separated.*

PROOF OF LEMMA 1. (A sketch—for details see [**F** and **W**].) Let $\mathcal{C}$ be an arbitrary nice chain.

Suppose X is normal. For each $\beta < \mathrm{cf}\,\kappa$, enumerate the functions u from A_β to $\mathfrak{T}$, where $u(y)$ is in a fixed small neighborhood base of y, as $\{u_\beta^\delta\colon \delta < \kappa_\beta^+\}$. (This is the only use of GCH and the character of X being less than κ). Inductively define two disjoint closed subsets H and K of Y. At stage (β, δ), if possible, ruin every extension of u_β^δ from defining disjoint open sets separating H and K. Having defined H and K, use normality to separate them and define a neighborhood assignment U. For each β, why was not $U\,|\,A_\beta$ ruined? It must have happened that $|\,S(\mathcal{C}, U, \beta)\,| \leq \kappa_\beta$. That is, U is thin w.r.t. $\mathcal{C}$.

Similarly for X countably paracompact, we must enumerate pairs (u, j) where $u\colon A_\beta \to \mathfrak{T}$ and $j\colon A_\beta \to \omega$, and we must define a partition $\{H_i\colon i < \omega\}$ of Y. For X $\aleph_1$-paralindelöf, $j\colon A_\beta \to \omega_1$, and the partition of Y is $\{H_i\colon i < \omega_1\}$. $\square$

We need some preparation for Lemma 2. Given a nice chain $\mathcal{C}$, we define $b\colon Y \to \mathrm{cf}\,\kappa$ by $b(y) = \min\{\beta < \kappa\colon y \in A_{\beta+1}\}$. If $\mathcal{C}$ has a prime or subscript, then the b defined from $\mathcal{C}$ has the same.

LEMMA 3. *If Y is sparse, let $\mathcal{C}$ be a nice chain and U a neighborhood assignment w.r.t. $\mathcal{C}$. Abbreviate $S(\mathcal{C}, U, \beta)$ by S_β. There is a nice chain $\mathcal{C}'$ and a neighborhood assignment U' satisfying*:

(i) *for all $\beta < \mathrm{cf}\,\kappa$, $A'_\beta \supset A_\beta \cup S_\beta$;*
(ii) *if $y \notin S_\beta$, then $b'(y) = b(y)$;*
(iii) *if $y \in S_\beta$, then $b'(y) < b(y)$;*
(iv) *for all y, z, if $b(z) < b'(y)$, then $U(z) \cap U'(y) = \varnothing$.*

PROOF. We would like to simply set $A'_\beta = A_\beta \cup S_\beta$, but then $A'_\lambda = \bigcup\{A'_\beta\colon \beta < \lambda\}$ might fail. So for limit ordinals γ less than $\mathrm{cf}\,\kappa$, let $(T_\beta^\gamma)_{\beta<\gamma}$ be a nice chain for S_γ. Precisely, $\bigcup\{T_\beta^\gamma\colon \beta < \gamma\} = S_\gamma$; if $\beta < \gamma$, then $|\,T_\beta^\gamma\,| \leq \kappa_\beta$; if $\alpha < \beta < \gamma$, then $T_\alpha^\gamma \subset T_\beta^\gamma$; and for limit ordinals λ, $\bigcup\{T_\beta^\gamma\colon \beta < \lambda\} = T_\lambda^\gamma$. Set

$$A'_\beta = A_\beta \cup S_\beta \cup \left(\bigcup \left\{T_\beta^\gamma\colon \gamma < \mathrm{cf}\,\kappa, \gamma \text{ a limit}\right\}\right).$$

(Here is where the fact that κ is singular is used. A'_β is the union of $\mathrm{cf}\,\kappa < \kappa_\beta$ many sets of cardinality no greater than κ_β.)

Let $y \in Y$ be arbitrary. Let β be least such that $y \in S_\beta$ (if any exist). Then $y \in A'_\beta$; hence $b'(y) < \beta$. We have shown that for all y, $y \notin S_{b'(y)}$. That is, $y \notin \bigcup \{U(z): b(z) < b'(y)\}$. Hence a neighborhood assignment U' satisfying (iv) can be defined. $\square$

PROOF OF LEMMA 2. We define nice chains $\mathcal{C}_i$ and neighborhood assignments U_i, U'_i by induction on $i < \omega$. Let $\mathcal{C}_0$ be arbitrary. If $\mathcal{C}_i$ has been defined, by sparseness, choose U_i thin w.r.t. to $\mathcal{C}_i$. Apply Lemma 3 to $\mathcal{C}_i$, U_i to get $\mathcal{C}'_i$ and U'_i. Set $\mathcal{C}_{i+1} = \mathcal{C}'_i$. By $< \kappa$-separated, define a neighborhood assignment U''_i so that for each $\beta < \operatorname{cf} \kappa$, $\{U_i(y): b_i(y) = \beta \text{ or } b_{i+1}(y) = \beta\}$ is disjoint.

For each $y \in Y$ and $i < \omega$, $b_{i+1}(y) \leqslant b_i(y)$; hence there is $n(y) < \omega$ so that for all $i \geqslant n(y)$, $b_i(y) = b_{n(y)}(y)$. We define a neighborhood assignment W: $Y \to \mathfrak{T}$ by

$$W(y) = \bigcap_{i \leqslant n(y)+1} (U_i(y) \cap U'_i(y) \cap U''_i(y)).$$

We claim that $\{W(y): y \in Y\}$ is disjoint. Let y, z be distinct elements of Y. Let $k = \min\{n(y), n(z)\}$. If $b_k(y) = b_k(z)$, then $U''_k(y) \cap U''_k(z) = \varnothing$. Without loss of generality, assume that $b_k(y) < b_k(z)$. If $b_k(y) < b'_k(z) = b_{k+1}(z)$, then $U_k(y) \cap U'_k(z) = \varnothing$. Hence $b_{k+1}(z) \leqslant b_k(z)$, and $k = n(y)$. If $b_{k+1}(z) = b_k(y)$, then $U''_k(y) \cap U''_k(z) = \varnothing$. So the only possibility left is $b_{k+1}(z) < b_k(y)$. Since $k = n(y)$, $b_{k+1}(z) < b_{k+2}(y) = b_k(y)$, hence $U'_{k+1}(y) \cap U_{k+1}(z) = \varnothing$. Hence $W(y) \cap W(z) = \varnothing$. $\square$

We say that a space is $\aleph_1$-paralindelöf if every open cover of cardinality ω_1 has a locally countable refinement.

COROLLARY. *Assume $V = L$. Discrete subsets of regular $\aleph_1$-paralindelöf spaces of character $\leqslant \omega_2$ can be separated.*

PROOF. By induction on κ, we prove that discrete sets of cardinality κ can be separated. For $\kappa = \omega_1$, we note that discrete sets in regular paralindelöf spaces can be separated by an open cover with the same cardinality as the discrete set. For other regular κ, Watson's proof [W] generalizes in a straightforward manner. For singular κ, first use Lemma 1 and then Lemma 2.

REFERENCES

[F] W. G. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. **46** (1974), 294–298.

[W] W. S. Watson, *Applications of set theory to topology*, Ph.D. thesis, Univ. of Toronto, 1982.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PENNSYLVANIA 15260