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LAGRANGE IDENTITY FOR POLYNOMIALS AND δ-CODES OF LENGTHS $7t$ AND $13t$

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ABSTRACT. It is known that application of the Lagrange identity for polynomials (see [1]) is the key to composing four-symbol δ-codes of length $(2s+1)t$ for $s = 2^a 10^b 26^c$ and odd $t \leqslant 59$ or $t = 2^d 10^e 26^f + 1$, where a, b, c, d, e and f are nonnegative integers. Applications of the Lagrange identity also lead to constructions of four-symbol δ-codes of length u for $u = 7t$ or $13t$. Consequently, new families of Hadamard matrices of orders $4uw$ and $20uw$ can be constructed, where w is the order of Williamson matrices. Related topics on zero correlation codes are also discussed.

1. Introduction. A sequence $V = (v_k)_s = (v_1, v_2, \ldots, v_s)$ of vectors v_k, where v_k is one of m orthonormal vectors $i_1, i_2, \ldots, i_m$ or their negatives, is said to be *an m-symbol δ-code of length s* if $v(j) = 0$ for $j \neq 0$, where $v(j) \equiv \sum_{k=1}^{s-j} v_k \cdot v_{k+j}$ is the nonperiodic autocorrelation function of V. Another characterization of V being an m-symbol δ-code of length s is that the associated polynomial

$$V(z) \equiv \sum_{k=1}^{s} v_k z^{k-1} = \sum_{j=1}^{m} P_j(z) i_j$$

(where $P_j(z) = \sum_{k=1}^{s} p_{jk} z^{k-1}$, $\sum_{j=1}^{m} |p_{jk}| = 1$ and all $p_{jk} \in \{0, 1, -1\}$) satisfy $\sum_{j=1}^{m} |P_j(z)|^2 = s$, for any z in the unit circle $K = \{z \in \mathbf{C}: |z| = 1\}$.

When $m = 4$ it is convenient to set $i_1 = (1,1,0,0)$, $i_2 = (1,-1,0,0)$, $i_3 = (0,0,1,1)$ and $i_4 = (0,0,1,-1)$ as four column vectors. All distinct i_j are obviously orthogonal to each other and have $|i_j| = \sqrt{2}$ for each j. Such a four-symbol δ-code of length s is called *a regular δ-code of length s* (abbreviated RD(s)). A sequence $V = (v_k)_s$ of normalized vectors v_k is called *a zero correlation code of length s* if $v(j) = 0$ for $j \neq 0$. *A regular zero correlation code of length z* (abbreviated RZC(s)) is a zero correlation code of length s with vectors $v_k = (q_k, r_k, s_k, t_k)$ such that $|q_k| + |r_k| + |s_k| + |t_k| = 2$ for each k, where all q_k, r_k, s_k, and $t_k \in \{0, 1, -1\}$. We note here that $|v_k| = \sqrt{2}$ for each k, and a regular δ-code is a regular zero correlation code in which all distinct vectors (excluding negatives) are orthogonal to each other.

The following theorem is proved in [1].

THEOREM 1 (LAGRANGE IDENTITY FOR POLYNOMIALS). *Let a, b, c, d, e, f, g, h be polynomials of z in K with real coefficients. Also let*

$$\begin{aligned} W &= -b'e + af' + cg + dh, & X &= a'e + bf' + dg' - ch', \\ Y &= -d'e - cf + ag' - bh, & Z &= c'e - df + bg + ah', \end{aligned} \tag{1}$$

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where $p' = p(z^{-1})$ *for* $p = p(z)$. *Then*

$$|W|^2 + |X|^2 + |Y|^2 + |Z|^2 = \left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)\left(|e|^2 + |f|^2 + |g|^2 + |h|^2\right).$$

Turyn base sequences for length $t = 2m + p$ (abbreviated TBS(t)) are four $(1, -1)$-sequences $(A, B; C, D)$, respectively, of lengths $m + p$ and m, pairwise, such that $A = (a_k)_{m+p}$, $B = (b_k)_{m+p}$, $C = (c_k)_m$ and $D = (d_k)_m$ satisfy $a(j) + b(j) + c(j) + d(j) = 0$ for $j \neq 0$.

TBS(t) exist for odd $t \leqslant 59$ or $t = 2^a 10^b 26^c + 1$, where a, b and c are nonnegative integers (see **[4, 5]**). We note here that, from given TBS(t), RD(t) can be found as follows: $(A, 0)$, $(B, 0)$, $(0, C)$ and $(0, D)$.

2. General results. We shall use $\bar{P} = -P = (-p_k)_n$ and $P' = (p_{n-k+1})_n$ to represent the negative and the reverse of a sequence $P = (p_k)_n$. It should be noted that, as polynomials,

$$P' = P(z^{-1}) = \sum_1^n p_k z^{1-k} = z^{1-n} \sum_1^n p_{n-k+1} z^{k-1}$$

for $P = P(z) = \sum_1^n p_k z^{k-1}$.

THEOREM 2. *Let* $(A, B; C, D)$ *be given* TBS(t). *Then the following* $(Q, R; S, T)$ *is* RD($7t$).

$$\begin{aligned} Q &= (q_k) = (\bar{A}, C; 0,0; A, D; 0,0; A, C; 0,0; \bar{B}', 0), \\ R &= (r_k) = (\bar{B}, D; 0,0; B, \bar{C}; 0,0; B, D; 0,0; A', 0), \\ S &= (s_k) = (0,0; A, \bar{C}; 0,0; \bar{B}, \bar{C}; 0,0; A, C; 0, \bar{D}'), \\ T &= (t_k) = (0,0; B, \bar{D}; 0,0; A, \bar{D}; 0,0; B, D; 0, C'). \end{aligned} \tag{2}$$

PROOF. In (2), each column vector (q_k, r_k, s_k, t_k) is obviously one of i_1, i_2, i_3, i_4, or their negatives. Consequently, it suffices to prove that $\sum_1^4 |P_j(z)|^2 = 7t$ for any z in K, where $P_1 = (Q + R)/2$, $P_2 = (Q - R)/2$, $P_3 = (S + T)/2$ and $P_4 = (S - T)/2$; or, equivalently, that

$$|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 14t \quad \text{for any } z \text{ in } K. \tag{2'}$$

First, we put $a = A$, $b = B'$, $c = -C$, $d = -D'$, $e = 1$, $f = z^{-2t} + z^{2t}$, $g = (z^{-2t} + 1 - z^{2t})z^M$ and $h = z^{4t-1}$, where $M = t - m = m + p$, in (1). We can easily prove (2′) from Theorem 1 by observing that $Q = Yz^{2t+M}$, $R = Z'z^{2t+M}$, $S = Wz^{3t}$ and $T = X'z^{3t}$;

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t,$$

$|e|^2 = |h|^2 = 1$ and $|f|^2 + |g|^2 = 5$ for any z in K.

The above construction of RD($7t$) is based on the fact that there are nice sequences, $\mathbf{e} = (1)$, $\mathbf{f} = (1, 0, 1)$, $\mathbf{g} = (1, 1, -1)$ and $\mathbf{h} = (1)$, satisfying $|\mathbf{e}|^2 + |\mathbf{f}|^2 + |\mathbf{g}|^2 + |\mathbf{h}|^2 = 7$ for any z in K; $e(z) = \mathbf{e}(z^{2t})$ and $f(z) = \mathbf{f}(z^{2t})z^{-2t}$ are symmetric, i.e., $e' = e(z^{-1}) = e$ and $f' = f$; also $e + f$ and $g = \mathbf{g}(z^{2t})z^{M-2t}$ fit well to each other. Similarly Theorem 3 below is based on the fact that there are nice sequences,

$\mathbf{e} = (1,0,1,0,1)$, $\mathbf{f} = (-1,0,1,0,-1,1)$ and $\mathbf{g} = (-1,1,1,1,1,-1)$, satisfying $|\mathbf{e}|^2 + |\mathbf{f}|^2 + |\mathbf{g}|^2 = 13$ for any z in K; $g = \mathbf{g}(z^t)z^{-5t/2}$ is symmetric; also, e, f (defined below) and g fit well.

THEOREM 3. *Let* $(A, B; C, D)$ *be a given* TBS(t). *Then the following* $(Q, R; S, T)$ *is* RD($13t$).

(3)

$$\begin{aligned}
Q &= (A, D'; \bar{A}, \bar{C}; \bar{A}, D'; \bar{A}, C; \bar{A}, D'; A, \bar{C}; 0, C; 0,0; 0,0; 0,0; 0,0; 0,0; 0,0),\\
R &= (\bar{B}, C'; B, D; B, C'; B, \bar{D}; B, C'; \bar{B}, D; 0, \bar{D}; 0,0; 0,0; 0,0; 0,0; 0,0; 0,0),\\
S &= (0,0; 0,0; 0,0; 0,0; 0,0; 0,0; \bar{A}, 0; A, C; B', \bar{C}; \bar{A}, \bar{C}; B', \bar{C}; A, \bar{C}; B', C),\\
T &= (0,0; 0,0; 0,0; 0,0; 0,0; 0,0; B, 0; \bar{B}, \bar{D}; A', D; B, D; A', D; \bar{B}, D; A', \bar{D}).
\end{aligned}$$

PROOF. In (3) it is obvious that each (q_k, r_k, s_k, t_k) is one of the i_j's or their negatives. Therefore it is sufficient to prove that

$$|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 26t \quad \text{for any } z \text{ in } K. \tag{3'}$$

We put $a = A$, $b = B$, $c = C$, $d = D$, $e = (z^{-2t} + 1 + z^{2t})z^{t/2-1} = \mathbf{e}(z^t)z^{-3t/2-1}$, $f = \mathbf{f}(z^t)z^{t/2-m}$ and $g = \mathbf{g}(z^t)z^{-5t/2}$ in (1). We can prove (3′) without difficulty from Theorem 1 by observing that $Q = -Yz^{5t/2}$, $R = Zz^{5t/2}$, $S = -Wz^{19t/2+M}$ and $T = Xz^{19t/2+M}$, where $M = t - m = m + p$.

Golay complementary sequences of length s (abbreviated GCL(s)) are two $(1,-1)$-sequences of length s, $\mathbf{f} = (f_k)_s$ and $\mathbf{g} = (g_k)_s$, having $\mathbf{f}(j) + \mathbf{g}(j) = 0$ for $j \neq 0$ or, equivalently, the associated polynomials satisfy $|\mathbf{f}|^2 + |\mathbf{g}|^2 = 2s$ for any z in K (see [**3**]). The following theorem is somewhat similar to Theorem 2 of [**1**].

THEOREM 4. *Let* $(A, B; C, D)$ *be* TBS(t) *and* $(f_k)_s$ *and* $(g_k)_s$ GCL(s), *where* $s \geqslant 1$. *Then the following* (Q, R, S, T) *is* RZC($(2s+1)t$).

$$\begin{aligned}
Q &= (Af_s, Cg_1; Af_{s-1}, Cg_2; \ldots; Af_1, Cg_s; 0, D; 0,0; 0,0; \ldots; 0,0),\\
R &= \big(Ag_s, \bar{C}f_1; Ag_{s-1}, \bar{C}f_2; \ldots; Ag_1, \bar{C}f_s; \bar{B}, 0; 0,0; 0,0; \ldots; 0,0\big),\\
S &= \big(0,0; 0,0; \ldots; 0,0; 0, \bar{C}; Bf_s, Dg_s; Bf_{s-1}, Dg_{s-1}; \ldots; Bf_1, Dg_1\big),\\
T &= \big(0,0; 0,0; \ldots; 0,0; A, 0; Bg_1, \bar{D}f_1; Bg_2, \bar{D}f_2; \ldots; Bg_s, \bar{D}f_s\big).
\end{aligned} \tag{4}$$

PROOF. In (4) each (q_k, r_k, s_k, t_k) has $\sqrt{2}$ as its norm. First we put $a = A$, $b = B$, $c = C$, $d = D$, $e = 0$,

$$f = \sum_{k=-r}^{r-1} f_{r+k+1} z^{(2k+1)t/2+M}, \quad g = \sum_{k=-r}^{r-1} g_{r+k+1} z^{(2k+1)t/2},$$

where $r = s/2$, and $h = z^{(s+1)t/2}$ in (1). By observing that $Q = Wz^{(s-1)t/2+M}$, $R = Yz^{(s-1)t/2}$, $S = Xz^{(3s+1)t/2+M}$ and $T = Zz^{(3s+1)t/2}$, and by noting that $|A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t$, $|f|^2 + |g|^2 = 2s$ and $|h|^2 = 1$ for any z in K, we obtain, from Theorem 1,

$$\begin{aligned}
|Q|^2 + |R|^2 + |S|^2 + |T|^2 &= \big(|A|^2 + |B|^2 + |C|^2 + |D|^2\big)\big(|f|^2 + |g|^2 + |h|^2\big)\\
&= 2t(2s+1).
\end{aligned}$$

Let

(4′) $\mathbf{Q} = (\mathbf{A}, 0), \mathbf{R} = (\mathbf{B}, 0), \mathbf{S} = (0, \mathbf{C})$ and $\mathbf{T} = (0, \mathbf{D})$ be RD(u),

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are $(0, \pm 1)$-sequences such that (U, V) is a $(1, -1)$-sequence for $U = \mathbf{A}$ or $\mathbf{B}$, and $V = \mathbf{C}$ or $\mathbf{D}$.[1]

We note here that in Theorem 4, by replacing TBS(t): $(A, B; C, D)$ with RD(u) of (4′) and t with u in RZC, our argument is still valid. Consequently we have

THEOREM 4′. *Let* $((A, 0), (B, 0); (0, C), (0, D))$ *be* RD(u) *as defined in* (4′). *Then* (4) *is* RZC($(2s + 1)u$).

Four $(1, -1)$-sequences (E, F, G, H) of length h are said to be complementary if $e(j) + f(j) + g(j) + h(j) = 0$ for $j \neq 0$ or, equivalently, if $|E|^2 + |F|^2 + |G|^2 + |H|^2 = 4h$ for any z in K. We obtain such complementary sequences (E, F, G, H) from Theorem 4′, where $E = Q + T$, $F = Q - T$, $G = R + S$ and $H = R - S$.

COROLLARY 1. *There exist four complementary* $(1, -1)$*-sequences of length*

$$h = (2s + 1)(2g + 1)t,$$

where $s = 2^a 10^b 26^c$, $g = 3, 6$ *or* $2^d 10^e 26^f$, *and* $t \leqslant 59$ *or* $t = 2^i 10^j 26^k + 1$; a, b, c, d, e, f, i, j, *and* k *are nonnegative integers.*

COROLLARY 2. *There exist Goethals-Seidel (Hadamard) matrices of order* $4h$, *where* h *is defined in Corollary* 1.

Although we have not found nice $(0, \pm 1)$-sequences for composing RD($11t$), RZC($11t$) can be constructed as follows.

THEOREM 5. *Let* $(A, B; C, D)$ *be* TBS(t). *Then the following* (Q, R, S, T) *is* RZC($11t$).

$$\begin{aligned}
Q &= (A, C; \bar{A}, \bar{C}; \bar{B}', \bar{C}; \bar{A}, C; 0, D; 0, D; 0, D; 0, 0; 0, 0; 0, 0; 0, 0), \\
R &= (\bar{A}, \bar{C}; A, D'; A, \bar{C}; \bar{A}, C; B, 0; B, 0; B, 0; 0, 0; 0, 0; 0, 0; 0, 0), \\
S &= (0, 0; 0, 0; 0, 0; 0, 0; A, 0; A, 0; A, 0; B, D; \bar{B}, C'; \bar{B}, D; B, \bar{D}), \\
T &= (0, 0; 0, 0; 0, 0; 0, 0; 0, C; 0, C; 0, C; \bar{B}, \bar{D}; B, D; \bar{A}', D; B, \bar{D}).
\end{aligned} \tag{5}$$

PROOF. In (5), each (q_k, r_k, s_k, t_k) has the same norm $\sqrt{2}$. It is known that the four sequences $\mathbf{e} = (1)$, $\mathbf{f} = (-1, 0, -1, 1)$, $\mathbf{g} = (1, -1, -1, 1)$ and $\mathbf{h} = (1, 1, 1)$ satisfy $|\mathbf{e}|^2 + |\mathbf{f}|^2 + |\mathbf{g}|^2 + |\mathbf{h}|^2 = 11$ for any z in K. In (1), we let $e = z^{t/2-1}$, $f = \mathbf{f}(z^t)z^{-3t/2+M}$, $g = \mathbf{g}(z^t)z^{-3t/2}$ and $h = \mathbf{h}(z^t)z^{5t/2}$. Also let $p = P$ for $p = a, b, c$ and d. By observing that $Q = Wz^{3t/2+M}$, $R = -Yz^{3t/2}$, $S = Zz^{17t/2}$ and $T = -Xz^{17t/2+M}$, we obtain from Theorem 1, $|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 22t$ for any z in K.

From Theorem 5 we also obtain four complementary sequences of length $11t$: $E = Q + S$, $F = Q - S$, $G = R + T$ and $H = R - T$.

[1] E.g. in RD($3t$), $(\mathbf{A}, 0) = (A, C; 0, 0; \bar{B}', 0)$, $(\mathbf{B}, 0) = (B, D; 0, 0; A', 0)$, $(0, \mathbf{C}) = (0, 0; A, \bar{C}; 0, \bar{D}')$ and $(0, \mathbf{D}) = (0, 0; B, \bar{D}; 0, C')$; thus $(\mathbf{A}, \mathbf{C}) = (A, C; A, \bar{C}; \bar{B}', \bar{D}')$, etc.

COROLLARY 3. *There exist four complementary* $(1, -1)$*-sequences of length* $11t$.

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