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A REMARK ON HORROCKS' THEOREM ABOUT PROJECTIVE $A[T]$-MODULES

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ABSTRACT. A projective module P over $A[T]$ needs no more generators than P_f for a monic f, and it is nearly determined by P_f.

The rings we consider here are assumed to be commutative with identity and the modules finitely generated. For a module M over a ring R, we let $\mu(M)$ denote the least number of elements in M required to generate M as an R-module.

Horrocks' Theorem [**1**] says the following:

THEOREM 1. *Let A be a local ring and let P be a projective $A[T]$-module. If P_f becomes a free $A[T]_f$-module, for some monic polynomial f in $A[T]$, then P is free. In other words, if* $\mu(P_f) = \operatorname{rank}(P_f)$ $(= \operatorname{rank}(P))$, *then* $\mu(P) = \operatorname{rank}(P)$.

By coupling Horrocks' Theorem with a recent result of Amit Roy [**3**, Theorem 1.1], one can even prove the following "qualitative" version of the Horrocks Theorem.

THEOREM 1′. *Let A be a local ring and let P be a projective $A[T]$-module. Then, for any monic polynomial f in $A[T]$, the $A[T]$-module P and the $A[T]_f$-module P_f have the same minimal number of generators, i.e.* $\mu(P) = \mu(P_f)$.

Let us first record the result of Amit Roy and then the proof of Theorem 1′ will follow immediately.

THEOREM 2. *Let A be a local ring and let P and Q be projective $A[T]$-modules. Assume that* $\operatorname{rank}(P) < \operatorname{rank}(Q)$. *Suppose that P_f is a direct summand of Q_f for some monic polynomial f in $A[T]$. Then P is a direct summand of Q.*

PROOF OF THEOREM 1′. Let f be a monic polynomial in $A[T]$. We know that $\mu(P_f) \leqslant \mu(P)$. So we need to prove that $\mu(P) \leqslant \mu(P_f)$. If P_f is a free $A[T]_f$-module then P is free by Horrocks' Theorem, and hence we get the desired conclusion. So we assume that P_f is not free. Then $\operatorname{rank}(P_f) < \mu(P_f)$. Let $\mu(P_f) = n$. We can present P_f as a direct summand of $(A[T]_f)^n \cong (A[T]^n)_f$. Since $\operatorname{rank}(P) = \operatorname{rank}(P_f) < \operatorname{rank}(A[T]^n)$, Amit Roy's Theorem is applicable to give that P is a direct summand of $A[T]^n$. Hence $\mu(P) \leqslant n$. Q.E.D.

We are naturally led to expect the following to be true.

Question. Let A be a local ring and let P and Q be projective $A[T]$-modules. Suppose that $P_f \cong Q_f$ for some monic polynomial f in $A[T]$. Is $P \cong Q$?

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Let us recall a couple of definitions.

DEFINITION. Let R be a ring and let P and P' be projective R-modules. Then P and P' are called *stably isomorphic* if $P \oplus R^n \cong P' \oplus R^n$ for some integer $n \geqslant 0$.

DEFINITION. Let R be a ring and let P be a projective R-module. We say that P satisfies the *cancellation condition* if any P' stably isomorphic to P is in fact isomorphic to P.

Regarding the above question, in [3] it had been pointed out that P and Q are "fairly close" to being isomorphic. We shall list here some easy consequences.

(1) If P or Q is free then by Theorem 1, $P \cong Q$.

(2) P is a direct summand of $Q \oplus Q'$ for any nonzero projective $A[T]$-module Q'.

PROOF. A direct consequence of Theorem 2.

(3) P and Q are stably isomorphic.

PROOF. Let $Q \oplus Q'$ be a free $A[T]$-module for a suitable Q'. Then $(P \oplus Q')_f$ is isomorphic to $(Q \oplus Q')_f$ and hence $P \oplus Q'$ is free by (1). If $Q' \oplus Q'' \cong A[T]^n$ then it follows that $P \oplus A[T]^n \cong Q \oplus A[T]^n$.

(4) If P or Q contains a rank 1 direct summand then $P \cong Q$.

PROOF. Suppose $P = P' \oplus L$ where L is of rank 1. Since $P_f \cong Q_f$, we get that P'_f is a direct summand of Q_f. Moreover, $\operatorname{rank}(P') < \operatorname{rank}(P) = \operatorname{rank}(Q)$. Therefore P' is a direct summand of Q, by Theorem 2. Say $Q \cong P' \oplus L'$. Now, as P and Q are stably isomorphic we obtain that L and L' are stably isomorphic. But it is well known that rank 1 projective modules satisfy the cancellation condition. Therefore $L \cong L'$ and hence $P \cong Q$.

(5) If $\operatorname{rank}(P) = 1$ then $P \cong Q$.

PROOF. Contained in (4).

(6) $P \oplus L \cong Q \oplus L$ for any rank 1 projective $A[T]$-module L.

PROOF. Consequence of (4).

(7) P and Q have the same minimal number of generators.

PROOF. Similar to that given for Theorem 1$'$.

The recent work of Plumstead [2] allows us to prove that the above question has a positive answer if A is noetherian of dimension $\leqslant 1$. In fact, for this situation one does not need A to be local. We first cite a theorem of Plumstead.

THEOREM 3. *Let R be a noetherian polynomial ring of dimension d. Then every projective R-module of rank $\geqslant d$ satisfies the cancellation condition.*

Now let A be a noetherian ring and let $R = A[T]$. Let P and Q be projective R-modules such that $P_f \cong Q_f$ for some monic polynomial f in R. We can appeal to the Quillen-Suslin affine Horrocks' Theorem and conclude that P and Q are stably isomorphic. If $\dim(A) \leqslant 1$ or equivalently $\dim(R) \leqslant 2$, then P and Q would be isomorphic, by Theorem 3 and the fact that rank 1 projectives satisfy the cancellation condition.

This remark came into existence out of a discussion with Paul Eakin. In fact Paul Eakin had, several years ago, posed Theorem 1$'$ as a question.

It would be interesting to prove Theorem 1$'$ without the assumption that A is a local ring.

References

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3. Amit Roy, *Remarks on a result of Roitman*, Tata Institute, preprint.

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