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FINITE GENERATION OF NOETHERIAN GRADED RINGS

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ABSTRACT. Let H be an additive abelian group. Then a commutative ring A is said to be H-graded if there is given a family $\{A_h\}_{h\in H}$ of subgroups of A such that $A = \oplus_{h\in H} A_h$ and $A_h A_g \subset A_{h+g}$ for all $h, g \in H$. In this note it is proved, provided H is finitely generated, that an H-graded ring A is Noetherian if and only if the ring A_0 is Noetherian and the A_0-algebra A is finitely generated. This is not true in general unless H is finitely generated. A counterexample is given.

1. Introduction. Let $A = \oplus_{s\geq 0} A_s$ be a commutative graded ring. Then as is well known, A is a Noetherian ring if and only if the ring A_0 is Noetherian and the A_0-algebra A is finitely generated. In this note we would like to expand this result to the case of more general graded rings. Let H be an additive abelian group and A a commutative ring. Then A is said to be H-graded if there is given a family $\{A_h\}_{h\in H}$ of subgroups of A such that

$$A = \bigoplus_{h\in H} A_h \quad \text{and} \quad A_h A_g \subset A_{h+g} \quad \text{for all } h, g \in H.$$

Notice that in case A is H-graded, A_0 (resp. A_h ($h \in H$)) is a subring (resp. an A_0-submodule) of A. With this terminology our expansion is stated as follows.

THEOREM (1.1). *Let A be an H-graded ring. Assume H is finitely generated. Then the following are equivalent*:

(1) *A is a Noetherian ring*;

(2) *the ring A_0 is Noetherian and the A_0-algebra A is finitely generated*;

(3) *every graded ideal of A is finitely generated.*

As an immediate consequence of (1.1), we have

COROLLARY (1.2). *Let S be a submonoid of a finitely generated abelian group and k a commutative ring. Then the following are equivalent*:

(1) *the monoid algebra $k[S]$ of S over k is Noetherian*;

(2) *k is Noetherian and S is finitely generated.*

We shall prove Theorem (1.1) in §2. Unless H is finitely generated, Theorem (1.1) is not true in general. A counterexample is given in §3.

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2. Proof of Theorem (1.1). We begin with

LEMMA (2.1). *Let B be a commutative ring and I an ideal of a subring A of B. Suppose A is a direct summand of B as an A-module. Then I is finitely generated if and only if the ideal IB of B is finitely generated.*

PROOF. It suffices to prove the *if part*. Choose elements $x_1, x_2, \ldots, x_r$ of I so that $IB = (x_1, x_2, \ldots, x_r)B$. Let $a \in I$. We write $a = \sum_{i=1}^{r} x_i b_i$ with $b_i \in B$. Let $\phi\colon B \to A$ be an A-linear map such that $\phi(a) = a$ for all $a \in A$. Then as $a, x_i \in A$, we get $a = \sum_{i=1}^{r} x_i \cdot \phi(b_i)$ in A, whence $a \in (x_1, x_2, \ldots, x_r)A$. Thus $I = (x_1, x_2, \ldots, x_r)A$.

LEMMA (2.2). *Let H be an abelian group and A an H-graded ring. Let $h \in H$ and suppose the ideal $A_h \cdot A$ of A is finitely generated. Then the A_0-module A_h is finitely generated.*

PROOF. Choose elements $x_1, x_2, \ldots, x_r$ of A_h so that $A_h \cdot A = (x_1, x_2, \ldots, x_r)A$. Let $x \in A_h$ and write $x = \sum_{i=1}^{r} x_i y_i$ with $y_i \in A$. We denote by y_{i0}, for each $1 \leqslant i \leqslant r$, the homogeneous component in y_i of degree 0. Then as $x, x_i \in A_h$, we get $x = \sum_{i=1}^{r} x_i \cdot y_{i0}$, whence $x \in \sum_{i=1}^{r} x_i A_0$. Thus $A_h = \sum_{i=1}^{r} x_i A_0$.

Let $B = \bigoplus_{s \in \mathbf{Z}} B_s$ denote a $\mathbf{Z}$-graded ring. We put $B_+ = \bigoplus_{s>0} B_s$ and $B_- = \bigoplus_{s<0} B_s$.

LEMMA (2.3). (1) *Suppose every ideal of B generated by elements of B_0 is finitely generated. Then the ring B_0 is Noetherian.*

(2) *Assume all the ideals $B_+ \cdot B$, $B_- \cdot B$, and $B_s \cdot B$ $(s \in \mathbf{Z})$ of B are finitely generated. Then the B_0-algebra B is finitely generated.*

PROOF. (1) Let I be an ideal of B_0. Then by the assumption, the ideal IB of B is finitely generated, whence by (2.1) I is. Thus B_0 is Noetherian.

(2) Let $f_1, f_2, \ldots, f_r$ (resp. $g_1, g_2, \ldots, g_t$) be homogeneous elements of B with positive (resp. negative) degree such that $B_+ \cdot B = (f_1, f_2, \ldots, f_r)B$ (resp. $B_- \cdot B = (g_1, g_2, \ldots, g_t)B$). Let $d = \max\{\deg f_i | 1 \leqslant i \leqslant r\}$ and $e = \min\{\deg g_i | 1 \leqslant i \leqslant t\}$. We put $C = B_0[B_s | e \leqslant s \leqslant d]$.

Claim. $C = B$.

First, notice that $C \supset B_s$ $(0 \leqslant s \leqslant d)$. Let $k > d$ be an integer and assume that $C \supset B_s$ for all $0 \leqslant s < k$. Let $x \in B_k$. Then as $x \in B_+$, we may write $x = \sum_{i=1}^{r} f_i x_i$ with $x_i \in B_{k-\deg f_i}$ $(1 \leqslant i \leqslant r)$. Recall that $k > k - \deg f_i \geqslant 0$ and we get, by the assumption on k, that $C \ni x_i$ for all $1 \leqslant i \leqslant r$. Hence, $C \ni x$ as $C \ni f_i$ by definition. Thus $C \supset B_+$. We similarly get that $C \supset B_-$, whence $C = B$ as claimed.

Since all the B_0-modules B_s $(e \leqslant s \leqslant d)$ are, by (2.2), finitely generated, we conclude that the B_0-algebra $B = B_0[B_s | e \leqslant s \leqslant d]$ is finitely generated. This completes the proof of Lemma (2.3).

PROOF OF THEOREM (1.1). It suffices to show (3) $\Rightarrow$ (2).

Case 1. $H = \mathbf{Z}^n$. We shall prove this by induction on n. If $n = 0$, there is nothing to say. See (2.3) for $n = 1$. Assume $n \geqslant 2$ and our implication is true for $n - 1$. We put

$$B_s = \sum_{h \in \mathbf{Z}^n \text{ such that } |h| = s} A_h$$

for each $s \in \mathbf{Z}$, where $|h| = \sum_{i=1}^{n} h_i$ for $h = (h_1, h_2, \ldots, h_n) \in \mathbf{Z}^n$. Then the family $\{B_s\}_{s \in \mathbf{Z}}$ defines a $\mathbf{Z}$-graduation on A, and we denote A by B when we consider the ring A to be a $\mathbf{Z}$-graded ring with graduation $\{B_s\}_{s\in\mathbf{Z}}$.

Now notice that all the ideals $B_+ \cdot B$, $B_- \cdot B$, and $B_s \cdot B$ ($s \in \mathbf{Z}$) of B are finitely generated, as they are still graded with respect to the original H-graduation in A. Hence B is, by (2.3)(2), a finitely generated B_0-algebra.

Let $\phi\colon \mathbf{Z}^{n-1} \to \mathbf{Z}^n$ be the homomorphism of groups defined by $\phi(h) = (h, -\sum_{i=1}^{n-1} h_i)$ for each $h = (h_1, h_2, \ldots, h_{n-1}) \in \mathbf{Z}^{n-1}$. We put $C_h = A_{\phi(h)}$ ($h \in \mathbf{Z}^{n-1}$). Then it is easy to check that the family $\{C_h\}_{h\in\mathbf{Z}^{n-1}}$ defines a $\mathbf{Z}^{n-1}$-graduation on B_0. We denote by C the ring B_0 considered to be $\mathbf{Z}^{n-1}$-graded with this graduation $\{C_h\}_{h\in\mathbf{Z}^{n-1}}$.

Recall that any homogeneous element of C is homogeneous also in A. Then we find, by (3), that the ideal IA of A is finitely generated for any graded ideal I of C. Therefore every graded ideal I of C is, by (2.1), finitely generated, whence we get, by the hypothesis of induction on n, that the ring $C_0 = A_0$ is Noetherian and the C_0-algebra C is finitely generated. Thus the required assertion (2) follows, because the C-algebra A is finitely generated as is remarked earlier.

Case 2 (*general case*). Let T denote the torsion part of H. Then as H is finitely generated, we may write $H = \mathbf{Z}^n \oplus T$, where $n = \operatorname{rank}_{\mathbf{Z}} H$. We put

$$D_t = \bigoplus_{h\in\mathbf{Z}^n} A_{h+t}$$

for each $t \in T$. Notice that $\{D_t\}_{t\in T}$ is a T-graduation on A. We denote A by D when we regard A as a T-graded ring with this graduation. Then all the D_0-modules D_t ($t \in T$) are, by (2.2), finitely generated, since the ideals $D_t \cdot D$ of D are finitely generated by (3) (recall that the ideals $D_t \cdot D = \sum_{h\in\mathbf{Z}^n} A_{h+t} \cdot A$ are graded also in A). Therefore the ring D is a module-finite extension of D_0, since $D = \bigoplus_{t\in T} D_t$ and T is a finite set.

Now consider the $\mathbf{Z}^n$-graded ring $D_0 = \bigoplus_{h\in\mathbf{Z}^n} A_h$ and we see, by (2.1) and (3), that every graded ideal of D_0 is finitely generated. Therefore by Case 1, the ring $A_0 = [D_0]_0$ is Noetherian and the A_0-algebra D_0 is finitely generated. Thus assertion (2) follows, because the ring $A = D$ is a module-finite extension of D_0 as mentioned above. This completes the proof of Theorem (1.1).

Proof of Corollary (1.2). Let H be a finitely generated abelian group which contains S as a submonoid. We denote by t^h, for each $h \in S$, the canonical image of h in $A = k[S]$. Let

$$A_h = kt^h \quad (h \in S) \qquad \text{and} \qquad A_h = (0) \quad (h \notin S)$$

for $h \in H$. Then the family $\{A_h\}_{h\in H}$ defines an H-graduation on A and, as $k = A_0$, (1.2) follows from (1.1).

3. Example. Without the assumption that H is finitely generated, Theorem (1.1) is not true in general. More explicitly,

Proposition (3.1). *Let p be a prime number and $H = \mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$. Then there is an H-graded ring A which is a field and is an infinite algebraic extension of A_0.*

By this example one knows that condition (1) of Theorem (1.1) does not necessarily imply finite generation of A as an A_0-algebra.

PROOF. Let k be a field of characteristic p and X an indeterminate over k. Let K denote the algebraic closure of $k(X)$ and choose elements X_n ($n \geqslant 0$) of K so that

$$X_0 = X \quad \text{and} \quad X_{n+1}^p = X_n \quad \text{for all } n \geqslant 0.$$

We put $A = \bigcup_{n \geqslant 0} k(X_n)$. Let $h \in H$ and express

$$h = (a/p^s) \bmod \mathbf{Z} \tag{\#}$$

with integers $s > 0$ and a. We define

$$A_h = k(X) \cdot X_s^a.$$

Then A_h is determined only by h and does not depend on (#). It is easy to check that the family $\{A_h\}_{h \in H}$ is an H-graduation on A.

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