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A SPECTRAL CHARACTERIZATION OF UNIVERSALLY WEAKLY INNER AUTOMORPHISMS OF SEPARABLE C^*-ALGEBRAS

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ABSTRACT. Let A be a separable C^*-algebra, α a *-automorphism of A. The following theorem is proven: α is weakly inner in every faithful, nondegenerate representation of A if and only if α fixes each closed, two-sided ideal of A and the Borchers spectrum of each quotient automorphism vanishes.

1. Introduction. Let A be a C^*-algebra, α a *-automorphism of A. We say that α is *universally weakly inner* if for each nondegenerate, faithful representation π of A on a Hilbert space H, there is a unitary operator u in the closure of $\pi(A)$ relative to the weak operator topology for which $\pi(\alpha(a)) = u^*\pi(a)u$, $a \in A$. This is equivalent to the double transpose α'' of α being an inner automorphism of the enveloping von Neumann algebra A'' of A. Universally weakly inner automorphisms were first defined and studied by Kadison and Ringrose in [**4**], and have since then received the attention of numerous authors [**1, 2, 3, 5, 6, 10**]. In this note we will determine a necessary and sufficient condition for an automorphism of a separable C^*-algebra to be universally weakly inner.

Our result is in the same spirit as a theorem of George Elliott [**2**] on universally weakly inner automorphisms of GCR algebras. Elliott proved that an automorphism of a GCR algebra is universally weakly inner if and only if it fixes every closed two-sided ideal. As an example of Kadison and Ringrose shows [**4**, Example (a)], this does not hold for algebras which are not GCR, but our result shows that if the ideal-fixing hypothesis is augmented by requiring also that the Borchers spectrum of each quotient automorphism vanish, then this condition does determine universal weak innerness, at least for separable C^*-algebras (for a precise statement, see Theorem 2.3 of the following section).

In [**5**], A. Kishimoto solved an old problem of E. C. Lance [**6**, note added in proof, p. 688] by showing that a universally weakly inner automorphism of a separable, simple C^*-algebra A is inner in the multiplier algebra of A. Our theorem puts this important result of Kishimoto in its appropriate general context. We deduce our theorem from a beautiful result of Olesen and Pedersen on properly outer automorphisms [**8**, Theorem 6.6], a result which, using ideas of L. G. Brown, extends and refines the key ideas in Kishimoto's proof to arbitrary separable C^*-algebras.

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In what follows, $\mathrm{Aut}(A)$ will denote the group of all $*$-automorphisms of a C^*-algebra A, and when we say ideal, we will always mean a closed, two-sided ideal. We will identify the enveloping von Neumann algebra A'' of A with the σ-weak closure of the image of A in its universal representation.

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2. The results. We begin by recalling the definitions of the Connes and Borchers spectra of an automorphism (see §8.8 of [**9**] for a more general formulation of this). Let A be a C^*-algebra, $\alpha \in \mathrm{Aut}(A)$. Let $\mathcal{H}^\alpha(A)$ denote the set of nonzero, hereditary C^*-subalgebras of A which are fixed by α, and $\mathcal{H}_B^\alpha(A)$ denote those elements of $\mathcal{H}^\alpha(A)$ which generate essential ideals in A, i.e., ideals which intersect every other nonzero ideal of A nontrivially. The *Connes spectrum* $\Gamma(\alpha)$ *of* α is defined to be $\bigcap\{\sigma(\alpha|_C)\colon C \in \mathcal{H}^\alpha(A)\}$, where $\sigma(\alpha|_C)$ denotes the spectrum of $\alpha|_C$ considered as a bounded linear operator on C, and the *Borchers spectrum* $\Gamma_B(\alpha)$ *of* α is defined to be $\bigcap\{\sigma(\alpha|_C)\colon C \in \mathcal{H}_B^\alpha(A)\}$. Thus $\Gamma(\alpha) \subseteq \Gamma_B(\alpha)$ and $\Gamma(\alpha) = \Gamma_B(\alpha)$ if A is simple (in fact, as Olesen and Pedersen note in [**8**], $\Gamma(\alpha) = \Gamma_B(\alpha)$ if A is $\{\alpha\}'$-prime, where $\{\alpha\}' =$ commutant of $\{\alpha\}$ in $\mathrm{Aut}(A)$).

The theorem proved here will be a corollary of some results of [**8**]. The principal result of [**8**] characterizes a properly outer automorphism of a separable C^*-algebra in terms of ten equivalent conditions, two of which we will make use of here. For the convenience of the reader, we will isolate the conditions that we need in the following proposition.

2.1. PROPOSITION (CONDITIONS (vii) AND (xi) OF THEOREM 6.6 OF [**8**]). *Let A be a separable C^*-algebra. The following are equivalent*:

(i) *There is no $C \in \mathcal{H}^\alpha(A)$ for which $\alpha|_C = \exp\delta$ for some $*$-derivation δ of C*;

(ii) *There is no ideal I of A fixed by α for which $\alpha|_I$ is universally weakly inner* (*in* I'').

We begin with a lemma which relates the vanishing of the Borchers spectrum of an automorphism α to a condition which says that "locally" α must be close to the identity map.

2.2. LEMMA. *Let A be a C^*-algebra*, $\alpha \in \mathrm{Aut}(A)$. *Consider the following conditions on* α:

(i) $\Gamma_B(\alpha) = \{1\}$;

(ii) *For each $\varepsilon > 0$ and $C \in \mathcal{H}^\alpha(A)$, there exists $D \in \mathcal{H}^\alpha(A)$ with $D \subseteq C$ and* $\|(\alpha - \mathrm{id})|_D\| < \varepsilon$. *Then* (ii) $\Rightarrow$ (i), *and* (i) $\Rightarrow$ (ii) *if A is separable.*

PROOF. (ii) $\Rightarrow$ (i) (modelled on the proof of Theorem 4.3 of [**8**]). Let $\varepsilon > 0$. Let $\{B_i\}$ be a maximal family of elements of $\mathcal{H}^\alpha(A)$ such that the ideals $I(B_i)$ generated by the B_i's are pairwise orthogonal and such that $\|(\alpha - \mathrm{id})|_{B_i}\| < \varepsilon$, for each i. Let $C =$ norm closure of $\Sigma_i B_i$. Then $C \in \mathcal{H}^\alpha(A)$ and we claim that $I(C)$ is essential in A. Suppose not. Then there is a nonzero ideal J of A fixed by α which is orthogonal to $I(C)$. By (ii), there exists $D \in \mathcal{H}^{\alpha}(A)$ with $D \subseteq J$ and $\|(\alpha - \mathrm{id})|_D\| < \varepsilon$. Then $I(D) \subseteq J$ and is hence orthogonal to all $I(B_i)$'s, contradicting the maximality of $\{B_i\}$. Thus $C \in \mathcal{H}^{\alpha}_B(A)$.

Let $t \in \Gamma_B(\alpha)$. Then $t \in \sigma(\alpha|_C)$. But since

$$\|(\alpha - \mathrm{id})|_C\| = \sup_i \|(\alpha - \mathrm{id})|_{B_i}\| \leq \varepsilon,$$

$|t - 1| \leq \varepsilon$, and since ε is arbitrary, $t = 1$. Thus $\Gamma_B(\alpha) = \{1\}$.

Now suppose that A is separable (in fact one need only suppose that A has a countable approximate identity; see §3 of [**8**]). Assume that $\Gamma_B(\alpha) = \{1\}$. Let $\varepsilon > 0$ and $C \in \mathcal{H}^{\alpha}(A)$ be given. By Proposition 3.3 of [**8**], $\Gamma_B(\alpha|_C) \subseteq \Gamma_B(\alpha)$, and so $\Gamma_B(\alpha|_C) = \{1\}$. Now by Proposition 8.8.7 of [**9**], for each open neighborhood Ω of 1 in the unit circle T, there exists $D \in \mathcal{H}^{\alpha}(C)$ with $\sigma(\alpha|_D) \subseteq \Omega$. Hence for each $\eta > 0$, we can find $D \in \mathcal{H}^{\alpha}(C)$ for which the spectral radius of $(\alpha - \mathrm{id})|_D$ does not exceed η. Taking $\eta < \sqrt{3}$ and applying Theorem 8.7.7 of [**9**], we find $D \in \mathcal{H}^{\alpha}(C)$ and a *-derivation $\delta \colon D \to D$ with $\alpha|_D = \exp \delta$. δ and $\alpha|_D$ commute, whence by Lemma 4.1 of [**8**], there exists $D \in \mathcal{H}^{\alpha}(C) \subseteq \mathcal{H}^{\alpha}(A)$ with $\|(\alpha - \mathrm{id})|_D\| < \varepsilon$. Q.E.D.

We will say that $\alpha \in \mathrm{Aut}(A)$ is *locally close to the identity* if it satisfies condition (ii) of Lemma 2.2.

If I is an ideal of A which is fixed by $\alpha \in \mathrm{Aut}(A)$, we will denote by α/I the automorphism of A/I induced by α. We can now state and prove our theorem.

2.3. THEOREM. *Let A be a separable C^*-algebra*, $\alpha \in \mathrm{Aut}(A)$. *The following are equivalent*:

(i) *α is universally weakly inner*;

(ii) *For each proper ideal I of A which is fixed by α, α/I is locally close to the identity*;

(iii) *For each proper ideal I of A which is fixed by α, $\Gamma_B(\alpha/I) = \{1\}$*;

(iv) *α fixes each ideal of A and α/I is locally close to the identity for all proper ideals I of A*;

(v) *α fixes each ideal of A and $\Gamma_B(\alpha/I) = \{1\}$ for all proper ideals I of A.*

PROOF. It is well known and easy to see that any universally weakly inner automorphism fixes all ideals, so we need only prove the equivalence of (i), (ii), and (iii).

(i) ⇒ (ii). We may suppose with no loss of generality that $I = (0)$.

Let $\varepsilon > 0$, and let $C \in \mathcal{H}^{\alpha}(A)$. Let p be the open projection in A'' for which $C = A \cap pA''p$. Since α fixes C and C is σ-weakly dense in $pA''p$, α'' fixes p, and so p reduces u, whence $(\alpha|_C)'' = \alpha''|_{pA''p} = \mathrm{ad}(up)$. Thus $(\alpha|_C)''$ is inner in $C'' = pA''p$. Applying Proposition 2.1 to C, we find a $D' \in \mathcal{H}^{\alpha}(C)$ and a *-derivation δ of D' for which $\alpha|_{D'} = \exp \delta$, and so by Lemma 4.1 of [**8**], there is $D \in \mathcal{H}^{\alpha}(D') \subseteq \mathcal{H}^{\alpha}(C) \subseteq \mathcal{H}^{\alpha}(A)$ such that $\|(\alpha - \mathrm{id})|_D\| < \varepsilon$.

(ii) ⇒ (iii). Apply Lemma 2.2.

(iii) ⇒ (i). By Proposition 8.9.3 of [**9**], it suffices to prove that

$(*)$ for each $\varepsilon > 0$, there is an α''-fixed projection p in A'' with central cover equal to 1 such that $\|(\alpha'' - \mathrm{id})|_{pA''p}\| < \varepsilon$.

To this end we assert first that if $z < 1$ is a central projection in A'' which is fixed by α'', there is a nonzero, α''-fixed projection $q \leqslant 1 - z$ in A'' with $\|(\alpha'' - \mathrm{id})|_{qA''q}\| < \varepsilon$. Once this is established, a maximality argument similar to the one used in the proof of Lemma 2.2 will yield a projection $p \in A''$ which satisfies (∗).

By passing to a quotient of A, we may suppose that $z = 0$. We have $\Gamma_B(\alpha) = \{1\}$, and so by Lemma 2.2, there exists $C \in \mathcal{K}^\alpha(A)$ with

$$(**) \qquad \|(\alpha - \mathrm{id})|_C\| < \varepsilon.$$

If q denotes the open projection in A'' with $C = A \cap qA''q$, then q is fixed by α'', and the σ-weak closure of C in A'' is $qA''q$. It follows from (∗∗) that $\|(\alpha'' - \mathrm{id})|_{qA''q}\| \leqslant \varepsilon$, and we are done. Q.E.D.

Suppose now that A is simple. Then $\Gamma_B(\alpha) = \Gamma(\alpha)$, and Theorem 2.3 shows that α is universally weakly inner if and only if $\Gamma(\alpha) = \{1\}$. Combining this with Corollary 4.3 of [7], we deduce

2.4. COROLLARY (COROLLARY 2.3 OF [5]). *Let A be a separable, simple C^*-algebra with multiplier algebra $M(A)$. Then $\alpha \in \mathrm{Aut}(A)$ is universally weakly inner if and only if α is inner in $M(A)$.*

REMARK. One may wonder if the conditions of Theorem 2.3 can be replaced by the simpler condition that $\Gamma_B(\alpha) = \{1\}$. The following example shows that this is not possible.

Let H be an infinite-dimensional separable Hilbert space, and let K denote the C^*-algebra of compact operators on H. Let e be a projection on H with e equivalent to $1 - e$, and set $A = K + \mathbf{C}e + \mathbf{C}(1 - e)$. If u is any unitary on H with $ueu^* = 1 - e$, then $a \to uau^*$ defines an automorphism of A with $\Gamma_B(\alpha) = \{1\}$, but α is not universally weakly inner since α exchanges e and $1 - e$ while A/K is commutative.

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