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MINIMAL SURFACES WITH CONSTANT CURVATURE IN 4-DIMENSIONAL SPACE FORMS

Dedicated to Professor S. Sasaki on his 70th birthday

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ABSTRACT. We classify minimal surfaces with constant Gaussian curvature in a 4-dimensional space form without any global assumption. As a corollary of the main theorem, we show there is no isometric minimal immersion of a surface with constant negative Gaussian curvature into the unit 4-sphere even locally. This gives a partial answer to a problem proposed by S. T. Yau.

1. Introduction. Let $M^m(c)$ be a connected Riemannian m-manifold of constant sectional curvature c. In this paper we classify isometric minimal immersions of $M^2(K)$ into $M^4(c)$ without any global assumption and give some partial results for a classification of minimal immersions of $M^2(K)$ into $M^5(c)$.

Let $S^n(1)$ be the unit sphere in the $(n+1)$-dimensional Euclidean space R^{n+1}. The Clifford surface in $S^3(1)$ and the Veronese surface in $S^4(1)$ are the best known examples of minimal surfaces in space forms of positive curvature.

By a rigidity theorem of Calabi and DoCarmo and Wallach, Chern and Barbosa [**2, 5, 4, 1**], and a localization theorem of Wallach [**10**], a minimal surface with constant *positive* curvature in $S^4(1)$ is locally the totally geodesic $S^2(1)$ or the Veronese surface. The main theorem of this paper is

THEOREM 1. *Let $x : M^2(K) \to M^4(c)$ be an isometric minimal immersion of $M^2(K)$ into $M^4(c)$. If $K = c$, then x is totally geodesic. Otherwise, either*

(a) *$K = 0$, $c > 0$ and x is a locally Clifford surface in a 3-dimensional totally geodesic submanifold $M^3(c)$ of $M^4(c)$, or*

(b) *$K = c/3$, $c > 0$ and x is a locally Veronese surface in $M^4(c)$.*

As a corollary of the theorem, we show there is no isometric minimal immersion of the hyperbolic 2-plane, $H^2[-1]$, into $S^4(1)$ even locally. This gives a partial answer to problem 101 proposed by S. T. Yau [**12**, p. 692].

We cannot apply the method used to prove Theorem 1 to higher codimensional cases directly. However, in §4 we prove a nonexistence theorem for minimal

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immersions of $H^2[-1]$ into $S^5(1)$ under an additional condition on the 2nd fundamental form. Thus we still do not know whether $H^2[-1]$ can be minimally immersed in some $S^N(1)$ $(N \geqslant 5)$.

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2. Preliminaries. Let $x: M^2(K) \to M^5(c)$ be an isometric minimal immersion of $M^2(K)$ into $M^5(c)$. Let e_1, e_2, e_3, e_4, e_5 be local fields of orthonormal frames in $M^5(c)$ such that, restricted to $M^2(K)$, e_1 and e_2 are tangent to $M^2(K)$. Let w_i, $1 \leqslant i, j \leqslant 2$, and w_α, $3 \leqslant \alpha, \beta, \ldots \leqslant 5$, be the fields of dual frames of e_A, $1 \leqslant A, B, \ldots \leqslant 5$. The structure equations of $M^5(c)$ are given by $dw_A = \sum w_B \wedge w_{BA}$, $w_{AB} + w_{BA} = 0$, and $dw_{AB} = \sum w_{AC} \wedge w_{CB} - cw_A \wedge w_B$. The summation is taken for repeated indices. Restricting these frames to $M^2(K)$, we have $dw_{12} = -Kw_1 \wedge w_2$ and $w_\alpha = 0$, $3 \leqslant \alpha \leqslant 5$. Exterior differentiation of w_α gives $w_{i\alpha} = \sum h_{\alpha ij} w_j$ and $h_{\alpha ij} = h_{\alpha ji}$, where the $h_{\alpha ij}$'s are the components of the 2nd fundamental form of x. By the minimality of x, we find $h_{\alpha 11} + h_{\alpha 22} = 0$, $3 \leqslant \alpha \leqslant 5$. From these formulae, the Gauss equation is represented by

$$\sum_\alpha \left(h_{\alpha 11}^2 + h_{\alpha 12}^2\right) = c - K \geqslant 0. \tag{2.1}$$

Therefore $K = c$ occurs only when the immersion is totally geodesic. Hereafter we assume $c > K$. That is, the vector valued 2nd fundamental form $\sum_\alpha (\sum h_{\alpha ij} w_i \otimes w_j) e_\alpha$ does not vanish at any point of $M^2(K)$.

In this paper we use the theory of higher fundamental tensors developed in part 1 of [7]. Since Ogata pointed out that the proof of the rth $(\geqslant 3)$ order Codazzi equation in [7] is not correct, we use only results for 2nd fundamental tensors from [7]. However, the main theorems in [**7** and **8**] are valid.

We introduce some notation used in [7]. Let

$$K_{(2)} = \sum_\alpha \left(h_{\alpha 11}^2 + h_{\alpha 12}^2\right), \tag{2.2}$$

$$N_{(2)} = \left| \sum h_{\alpha 11} e_\alpha \wedge \sum h_{\alpha 12} e_\alpha \right|^2. \tag{2.3}$$

The nonnegative smooth function $N_{(2)}$ on $M^2(K)$ is the square of the area of the parallelogram generated by $\sum h_{\alpha 11} e_\alpha$ and $\sum h_{\alpha 12} e_\alpha$. Suppose $N_{(2)}$ is identically zero on $M^2(K)$. By a lemma of Otsuki [**9**, p. 96] (see also Lemma 2 in [7]), there exists a 3-dimensional totally geodesic submanifold $M^3(c)$ of $M^5(c)$ such that $x(M^2(K))$ is contained in $M^3(c)$ as a minimal surface. Then we have $c > 0$ and $K = 0$ by Chen [**3**, Corollary 1].

In case $N_{(2)}$ is not identically zero, the set $\Omega_2 = \{p \in M^2(K): N_{(2)}(p) \neq 0\}$ is open in $M^2(K)$. Since $\sum h_{\alpha 11} e_\alpha$ and $\sum h_{\alpha 12} e_\alpha$ are linearly independent at each point x of Ω_2, the 2nd osculating space $T_x^{(2)}$ is spanned by those vectors and e_i, $1 \leqslant i \leqslant 2$. We identify the first osculating space $T_x^{(1)}$, $x \in M^2(K)$, with the tangent space of $M^2(K)$. Let e_A be local orthonormal frame fields such that e_i, $1 \leqslant i \leqslant 2$, and e_α, $3 \leqslant \alpha \leqslant 4$, span $T_x^{(2)}$, $x \in \Omega_2$. Then on Ω_2 we have

$$W_{i5} = 0. \tag{2.4}$$

By taking the exterior derivative of (2.4) and using the structure equation of $M^5(c)$, we get $w_{i3} \wedge w_{35} + w_{i4} \wedge w_{45} = 0$. This allows us to introduce the quantities h_{5ijk} defined by the equations

$$\sum h_{\alpha ij} w_{\alpha 5} = \sum h_{5ijk} w_k. \tag{2.5}$$

The h_{5ijk}'s are symmetric in the Latin indices i, j, k and are called components of the 3rd fundamental form of $x_{|\Omega_2}$. By (2.5) and the minimality of x, we get $\Sigma h_{5iik} = 0$. Note that

$$h_{5ijk} = h_{5ij,k} \tag{2.6}$$

is easily verified by the definition of the covariant derivatives of $h_{\alpha ij}$, since for $\alpha \geqslant 3$,

$$Dh_{\alpha ij} = \sum h_{\alpha ij,k} w_k = dh_{\alpha ij} + \sum h_{\alpha sj} w_{si} + \sum h_{\alpha is} w_{sj} + \sum h_{\beta ij} w_{\beta\alpha}. \tag{2.7}$$

We set

$$K_{(3)} = h_{5111}^2 + h_{5112}^2. \tag{2.8}$$

It is easily verified that $K_{(3)}$ is an invariant of x restricted to Ω_2 under the fixed decomposition of the normal bundle, i.e., e_5 is always considered as a normal vector orthogonal to $T_x^{(2)}$. We put $f_{(2)} = K_{(2)}^2 - 4N_{(2)}$ and $H_\alpha = h_{\alpha 11} + ih_{\alpha 12}$. Then we have (cf. [7]) $f_{(2)} = |\Sigma H_\alpha^2|^2$ and the Codazzi equation implies

$$\left(d\sum H_\alpha^2 + 4i\left(\sum H_\alpha^2\right)w_{12}\right) \wedge (w_1 - iw_2) = 0. \tag{2.9}$$

Therefore, by Lemma 3 of [7]

$$\Delta \log f_{(2)} = 8K, \tag{2.10}$$

wherever $f_{(2)} \neq 0$, and we get

$$\Delta f_{(2)} = 8Kf_{(2)} + |Df_{(2)}|^2/f_{(2)}. \tag{2.11}$$

In general, [7] gives

$$\frac{1}{4}\Delta K_{(2)} = -2N_{(2)} + KK_{(2)} + K_{(3)} + \sum_{\alpha \leqslant 4} \left(h_{\alpha 11,1}^2 + h_{\alpha 11,2}^2\right). \tag{2.12}$$

By (2.1), (2.2) and (2.12), we obtain

$$|Dh_{311}|^2 + |Dh_{421}|^2 = 2N_{(2)} - K(c-K) - K_{(3)}. \tag{2.13}$$

3. Proof of Theorem 1. We assume $x(M^2(K))$ is contained in a totally geodesic submanifold $M^4(c)$ of $M^5(c)$. Then $K_{(3)}$ is identically zero by a lemma of Otsuki [**9**, p. 96]. By virtue of (2.13), we have

$$|Df_{(2)}|^2 = 8(c-K)^{-1} f_{(2)}^3 - 16(c-2K) f_{(2)}^2 + 8(c-K)^2(c-3K) f_{(2)}. \tag{3.1}$$

In fact, (3.1) is trivial if $f_{(2)}$ is identically zero. We assume $f_{(2)}$ is not zero. Then there exists an orthonormal frame e_A for which $h_{\alpha ij}$'s satisfy $h_{312} = h_{411} = 0$ and $h_{311} > h_{412} > 0$ on an open subset of Ω_2 (cf. Wong [**11**, p. 480]). With respect to these frame fields we have

$$K_{(2)} = h_{311}^2 + h_{421}^2 = c - K = \text{constant}; \qquad N_{(2)} = h_{311}^2 h_{421}^2. \tag{3.2}$$

Therefore we have, by (3.2) and (2.13),

$$\begin{aligned}|DN_{(2)}|^2 &= 4h_{311}^2\left(h_{421}^2 - h_{311}^2\right)^2 |Dh_{311}|^2 \\ &= \frac{4h_{311}^2 h_{421}^2}{h_{311}^2 + h_{421}^2}\left\{\left(h_{421}^2 + h_{311}^2\right)^2 - 4h_{421}^2 h_{311}^2\right\}\left(2N_{(2)} - K(c-K)\right) \\ &= 4N_{(2)}\left(K_{(2)}^2 - 4N_{(2)}\right)\left(2N_{(2)} - K(c-K)\right)/(c-K).\end{aligned}$$

By definition of $f_{(2)}$ and (3.2), this implies (3.1).

LEMMA 1. *Let $x : M^2(K) \to M^4(c)$ be an isometric minimal immersion with $N_{(2)} \neq 0$ on $M^2(K)$. Then we have $K = c/3$ and $c > 0$. Locally x is the Veronese surface in $M^4(c)$.*

PROOF. If $f_{(2)}$ is identically zero on $M^2(K)$, then $\sum h_{\alpha 11} e_\alpha$ and $\sum h_{\alpha 12} e_\alpha$ are orthogonal and have the same nonzero length (cf. [**7**, p. 475]). By normalizing these vectors, we adopt them as a part of the basis of $T_x^{(2)}$, $x \in M^2(K)$. For these new frame fields, we have $h_{311} = h_{412} > 0$, $h_{312} = h_{411} = 0$. The Gauss equation implies

$$h_{311}^2 = h_{412}^2 = (c-K)/2 = \text{constant} > 0.$$

We have $Dh_{311} = dh_{311} = 0$ and $2w_{12} = w_{34}$, as seen by the formulas

$$Dh_{312} = h_{311}(2w_{12} - w_{34}) = h_{311,2}w_1 - h_{311,1}w_2 = 0.$$

Exterior differentiation of $2w_{12} = w_{34}$ gives $K = h_{311}h_{421}$. It follows from these formulas that $K = h_{311}^2$ is positive, and we can see $K = c/3$ by (2.13), which implies $c > 0$.

It is known that such an immersion represents locally the Veronese surface (cf. for instance [**11**, Theorem 4.2]).

Next we will show that the case of $f_{(2)} \neq 0$ cannot happen. By (2.11) and (3.1), we obtain

$$\Delta f_{(2)} = 8(c-K)^{-1} f_{(2)}^2 - 8(2c - 5K)f_{(2)} + 8(c-K)^2(c-3K). \tag{3.3}$$

If we set $\theta = f_{(2)}$, then (3.1) and (3.3) are expressed in the following way: $\Delta\theta = \phi(\theta)$, $|D\theta|^2 = f(\theta)$, where $\phi(\theta)$ and $f(\theta)$ are polynomials for θ with constant coefficients. It is then proved that θ must be a constant function. If θ is not constant then there exist local coordinates (θ, τ) on $M^2(K)$ such that the first fundamental form is

$$ds^2 = \left(d\theta^2 + \exp\left(2\int \phi f^{-1}\, d\theta\right) d\tau^2\right)/f(\theta)$$

and the Gauss curvature K satisfies

$$fK + (\phi - f')(\phi - \tfrac{1}{2}f') + f(\phi' - \tfrac{1}{2}f'') = 0, \tag{3.4}$$

where the prime denotes differentiation with respect to θ (cf. Eisenhart [**6**, p. 164]). The left-hand side of (3.4) is a polynomial in θ such that the coefficients of θ^3 and θ are $8(8c - 27K)/(c-K)$ and $8(c-K)^2(c-3K)(8c-3K)$, respectively. Since (3.4) is a nontrivial polynomial with constant coefficients, θ must be a constant. Therefore $f_{(2)}$ is a nonzero constant, which implies $K = 0$ by (2.10) and $f_{(2)} = c^2 - 4N_{(2)}$. By (3.3), we have $f_{(2)} = c^2$, which is a contradiction because we have assumed $N_{(2)} \neq 0$. This proves Lemma 1.

We prove Theorem 1 as follows: If $N_{(2)}$ is identically zero, then x is totally geodesic or we have $K = 0$ and $c > 0$. Moreover, in this case, $x(M^2(0))$ is contained in a 3-dimensional totally geodesic $M^3(c)$ as part of a Clifford surface [3].

When $N_{(2)} \not\equiv 0$ and $K_{(3)} \equiv 0$, we apply the above lemma to $\Omega_{(2)}$. Since $f_{(2)}$ is identically zero on $\Omega_{(2)}$, $N_{(2)}$ is a positive constant on $\Omega_{(2)}$. Since $\Omega_{(2)}$ is open and closed, we have $\Omega_{(2)} = M^2(K)$, because $M^2(K)$ is connected by definition. Hence Theorem 1 has been proved.

4. The 3rd fundamental form. In this section we study the case $K_{(3)} \not\equiv 0$, which means that the minimal immersion $x : M^2(K) \to M^5(c)$ is full. We define covariant derivation for the 3rd fundamental tensor (cf. [7]) by

$$(4.1) \quad \tilde{D}h_{5ijk} = \sum h_{5ijk;l} w_l = dh_{5ijk} + \sum h_{5sjk} w_{si} + \sum h_{5isk} w_{sj} + \sum h_{5ijs} w_{sk}.$$

Exterior differentiation of (2.5) gives

$$(4.2) \qquad \sum h_{5ijk;l} w_l \wedge w_k = -\sum h_{\alpha ij,l} w_{\alpha 5} \wedge w_l.$$

When we set $w_{\alpha 5} = \sum a_{\alpha j} w_j$, (4.2) is equivalent to

$$(4.2)' \qquad h_{5ijk;l} - h_{5ijl;k} = \sum h_{\alpha ij,l} a_{\alpha k} - \sum h_{\alpha ij,k} a_{\alpha l}.$$

By (2.5) we have

$$(4.3) \qquad \sum h_{\alpha ij} a_{\alpha k} = h_{5ijk}.$$

LEMMA 2. *Let $x : M^2(K) \to M^5(c)$ be a full isometric minimal immersion. Then the 3rd order Codazzi equations*

$$(4.4) \qquad h_{5ijk;l} = h_{5ijl;k}$$

hold if $N_{(2)}$ is constant on $M^2(K)$.

PROOF. $f_{(2)}$ is constant on $M^2(K)$. If $f_{(2)}$ is zero, then we take frame fields e_A such that $h_{311} = h_{412}$ and $h_{312} = h_{411} = 0$. In the case of $f_{(2)} \neq 0$, we take Y. C. Wong's frame, for which $h_{311} > h_{412}$, $h_{312} = h_{411} = 0$ (cf. [11, p. 480]). Since $N_{(2)}$ is constant, in both cases, h_{311} and h_{421} are also locally constant, so $Dh_{3ij} = Dh_{4ij} = 0$. It follows that the right-hand side of (4.2)′ vanishes. Q.E.D.

Hereafter we assume $N_{(2)}$ is positive constant on $M^2(K)$. $f_{(2)}$ is also constant. Since we have $Dh_{\alpha ij} = 0$, by (2.13), $K_{(3)}$ is constant. We take frame fields e_i locally for which h_{5112} vanishes on a neighborhood of $M^2(K)$. It shall be remarked that e_3 and e_4 are any vectors such that e_i, e_3 and e_4 are bases of $T_x^{(2)}$. Since $K_{(3)} = h_{5111}^2 + h_{5112}^2$ is a nonzero constant, h_{5111} is locally a nonzero constant. From the formulas $Dh_{5111} = 0$ and $Dh_{5112} = 3h_{5111}w_{12} = h_{5111;2}w_1 - h_{5111;1}w_2 = 0$, which is derived by Lemma 2, we get $w_{12} = 0$. It implies $K = 0$ so c is positive by (2.1). Therefore Lemma 2 in [8] holds, so x is locally a generalized Clifford surface of $M^5(c)$ described in [8]. Summarizing, we get this result.

THEOREM 2. *Let $x : M^2(K) \to M^5(c)$ be a full isometric minimal immersion with $N_{(2)}$ constant on $M^2(K)$. Then $K = 0$ and c is positive. Locally x is one of the generalized Clifford surfaces described in* [8].

REMARKS. (1) If the rth (≥ 3) fundamental tensors all satisfy the rth order Codazzi equation, then we can prove a local classification theorem for minimal immersions $M^2(K) \to M^m(c)$ by methods similar to those proving Theorem 1.

(2) Let $x: M^2(K) \to M^5(c)$ be any full isometric minimal immersion. Then by (2.13) we have $|Dh_{311}|^2 + |Dh_{421}|^2 \leq 2N_{(2)} - K(c-K)$. It follows that

$$(4.5) \qquad |Df_{(2)}|^2 \leq 8(c-K)^{-1} f_{(2)}^3 - 16(c-2K) f_{(2)}^2 + 8(c-K)^2(c-3K) f_{(2)},$$

$$(4.6) \qquad \Delta f_{(2)} \leq 8(c-K)^{-1} f_{(2)}^2 - 8(2c-5K) f_{(2)} + 8(c-K)^2(c-3K).$$

Under suitable assumptions on the topology of $M^2(K)$, the author conjectures that the $f_{(2)}$ must be constant.

REFERENCES

1. J. L. M. Barbosa, *On minimal immersions of S^2 into S^{2m}*, Trans. Amer. Math. Soc. **210** (1975), 75–106.

2. E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geom. **1** (1967), 111–125.

3. B. Y. Chen, *Minimal surfaces with constant Gauss curvature*, Proc. Amer. Math. Soc. **34** (1972), 504–508.

4. S. S. Chern, *On the minimal immersions of the two-sphere in a space of constant curvature*, Problems in Analysis, Princeton Univ. Press, Princeton, N. J., 1970, pp. 27–40.

5. M. DoCarmo and N. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math. (2) **95** (1971), 43–62.

6. L. P. Eisenhart, *An introduction to differential geometry*, Princeton Univ. Press, Princeton, N. J., 1947.

7. K. Kenmotsu, *On compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature*. I, II, Tôhoku Math. J. **25** (1973), 469–479; **27** (1975), 291–301.

8. ______, *On minimal immersions of R^2 into S^N*, J. Math. Soc. Japan **28** (1976), 182–191.

9. T. Otsuki, *Minimal submanifolds with m-index 2 and generalized Veronese surfaces*, J. Math. Soc. Japan **24** (1972), 89–122.

10. N. Wallach, *Extension of locally defined minimal immersions of spheres into spheres*, Arch. Math. **21** (1970), 210–213.

11. Y. C. Wong, *Contributions to the theory of surfaces in a 4-space of constant curvature*, Trans. Amer. Math. Soc. **59** (1946), 467–507.

12. S. T. Yau, *Seminar on differential geometry*, Ann. of Math. Studies, No. 102, Princeton Univ. Press, Princeton, N. J., 1982.

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