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A COUNTEREXAMPLE TO THE GENERALIZED BANACH THEOREM

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ABSTRACT. We show that it is consistent that the family of Borel maps of class 2 differs from the family of pointwise limits of Borel maps of class 1. This gives an answer to a question raised by W. G. Fleissner.

1. Let X, Y be metric spaces. For $0 \leqslant \alpha \leqslant \omega_1$, denote by $\Sigma_\alpha X$ the family of sets in X of additive class α. Define the Borel classes by

$$\psi_\alpha(X, Y) = \{f\colon X \to Y\colon \forall(G \in \Sigma_0 Y) f^{-1}[G] \in \Sigma_\alpha X\}$$

and the Banach classes for $1 \leqslant \alpha \leqslant \omega_1$ by

$$\phi_1^*(X, Y) = \psi_1(X, Y)$$

and

$$\phi_\alpha^*(X, Y) = \text{family of all limits of pointwise convergent sequences of maps from } \bigcup_{\beta<\alpha} \phi_\beta^*(X, Y) \qquad (\alpha > 1).$$

Correspondence between the Borel classes and the Banach classes is expressed by

BANACH THEOREM. *Let Y be a separable metric space. Then $\phi_\alpha^*(X, Y) = \psi_\alpha(X, Y)$ or $\psi_{\alpha+1}(X, Y)$ according as α is finite or infinite.*[1]

Recall that the inclusion $\phi_\alpha^*(X, Y) \subseteq \psi_\alpha(X, Y)$ or $\psi_{\alpha+1}(X, Y)$, according as α is finite or infinite, holds for any metric space Y.

Recently W. G. Fleissner [$\mathbf{F_1}$] introduced an axiom called Proposition P. He proved that Proposition P is consistent, assuming that ZFC + 'there exists a supercompact cardinal' is consistent and that it implies, among other things, the Banach Theorem for any metric space Y. Later Fleissner, Hansell and Junnila [**FHJ**] proved that Proposition P is also implied by the Product Measure Extension Axiom.

In [$\mathbf{F_1}$] the author asks whether it is consistent that $\phi_2^*(X, Y) \neq \psi_2(X, Y)$. In this note we show how the well-known Miller Theorem [**M**] can be used to give an affirmative answer to this question.

2. Recall that a Q set is an uncountable subset X of the real line $\mathbf{R}$ such that every subset of X belongs to $\Sigma_1 X$. It follows from the Miller Theorem (cf. [$\mathbf{F_2}$, Theorem 23]) that the following proposition is consistent with ZFC.

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[1]For more details about ϕ^*, ψ and the Banach Theorem, see [**H**].

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PROPOSITION. *There is a subset X of* $\mathbf{R}$ *with cardinality* ω_3 *such that every subset of* X *belongs to* $\Sigma_2 X$, *but every* Q *set contained in* X *has cardinality* $< \omega_3$.

THEOREM. *If* $|X|$ *is the set* X *with discrete metric, then* $f \in \psi_2(X, |X|) \setminus \phi_2^*(X, |X|)$ *for every injection* f *from* X *to* $|X|$.

PROOF. Clearly $f \in \psi_2(X, |X|)$. Suppose $f \in \phi_2^*(X, |X|)$. Then there is a sequence $\{f_n : n \in \omega\} \subseteq \psi_1(X, |X|)$ such that f is the pointwise limit of f_n. Put $A_n = \{x \in X: f_n(x) = f(x)\}$. Since $X = \bigcup \{A_n : n \in \omega\}$ there is $n_0 \in \omega$ such that the cardinality of A_{n_0} is ω_3. This implies A_{n_0} is not a Q set. Hence, since $f | A_{n_0}$ is an injection, $f | A_{n_0} \notin \psi_1(A_{n_0}, |X|)$. On the other hand, $f_{n_0} | A_{n_0} \in \psi_1(A_{n_0}, |X|)$ and $f_{n_0} | A_{n_0} = f | A_{n_0}$, contradiction.

REMARK 1. Similarly, Miller's Theorem implies that for any nonlimit $\alpha > 1$ there is a set X_α such that $\phi_\alpha^*(X_\alpha, |X_\alpha|) \neq \psi_\alpha(X_\alpha, |X_\alpha|)$ or $\psi_{\alpha+1}(X_\alpha, |X_\alpha|)$ according as α is finite or infinite.

REMARK 2. Fleissner proved in **[$\mathbf{F_1}$]** that under Proposition P for every function f from $\psi_1(X, Y)$, where X and Y are arbitrary metric and Banach spaces, respectively, there is a residual set T in X such that $f | T$ is continuous and he raised the question as to whether 'residual' can be replaced by 'dense'. Observe, however, that if f is a function from the rationals to $\mathbf{R}$ which has a discrete image, then $f \in \psi_1(\mathbf{Q}, \mathbf{R})$ and $f | T$ is not continuous for any dense subset T of $\mathbf{Q}$.

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