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HOMOGENEOUS CIRCLE-LIKE CONTINUA[1]

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ABSTRACT. We extend earlier work of Bing, Jones, Hagopian, and Rogers to give a complete classification of nondegenerate homogeneous circle-like continua as pseudo-arcs, solenoids, or solenoids of pseudo-arcs, by showing that solenoids of pseudo-arcs are unique.

The simple closed curve is obviously a homogeneous continuum. By taking an inverse limit of simple closed curves where the bonding maps are covering maps one obtains the solenoids, which are also seen to be homogeneous [**6**]—in fact topological groups. Except for the simple closed curve (which can be considered a solenoid with each bonding map the identity), all of the solenoids are nonplanar [**3**].

Bing [**1**] and Moise [**14**] have shown that the pseudo-arc is a homogeneous plane continuum. Bing [**2**] has further characterized the pseudo-arc as the only nondegenerate, homogeneous, chainable continuum. The pseudo-arc is also seen to be circularly chainable and so is another example of a homogeneous, circularly chainable continuum. Bing presented the pseudo-circle as a circularly chainable, nonchainable, hereditarily indecomposable, plane continuum, and as a candidate for another homogeneous continuum. Fearnley [**7**] and Rogers [**15**] showed that the pseudo-circle is not homogeneous, and that there is no homogeneous, circularly chainable, nonchainable, hereditarily indecomposable continuum (whether planar or otherwise).

Jones [**12**] has shown that every homogeneous, decomposable, plane continuum has a continuous decomposition into mutually homeomorphic, homogeneous, nonseparating plane continua such that the decomposition space is a simple closed curve. Combining this with the result that the pseudo-arc is homogeneous, Bing and Jones [**4**] constructed a homogeneous circle-like plane continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a simple closed curve. They called this continuum a circle of pseudo-arcs and showed that it is the only continuum with the above properties.

Combining the above results Burgess [**5**] showed that a nondegenerate, circularly chainable, plane continuum is homogeneous if and only if it is either a simple closed curve, a pseudo-arc or a circle of pseudo-arcs. Thus any other example of a homogeneous circle-like continuum would have to be nonplanar (as the solenoids are).

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Just as the solenoids can be obtained as inverse limits of simple closed curves with covering maps as bonding maps, Rogers [**16**] showed that one can take the inverse limit of circles of pseudo-arcs with covering maps as bonding maps to obtain homogeneous, circle-like continua with continuous decompositions into pseudo-arcs such that the decomposition spaces are solenoids. He called these continua solenoids of pseudo-arcs. For every inverse sequence of simple closed curves giving a solenoid there exists a corresponding inverse sequence of circles of pseudo-arcs giving a solenoid of pseudo-arcs with a decomposition which yields the given solenoid. Solenoids of pseudo-arcs with decompositions yielding distinct solenoids are nonhomeomorphic, and so there are $c = 2^{\omega_0}$ topologically distinct solenoids of pseudo-arcs, each of which is a homogeneous, circle-like continuum.

Hagopian [**9**] showed that the solenoids are characterized as the only homogeneous continua such that every nondegenerate proper subcontinuum of them is an arc. Hagopian and Rogers [**11**] then presented a classification of homogeneous circle-like continua. They defined a continuum M to be a solenoid of pseudo-arcs if M is circle-like and admits a continuous decomposition into pseudo-arcs with the decomposition space a solenoid, whether M is obtained as an inverse limit of circles of pseudo-arcs in the manner described by Rogers or not. They then showed that every nondegenerate homogeneous circle-like continuum is either a pseudo-arc, a solenoid, or a solenoid of pseudo-arcs (with the simple closed curve and the circle of pseudo-arcs considered as special cases of the solenoid and the solenoid of pseudo-arcs respectively).

There were, however, a couple of questions left unanswered by their work. Rogers had shown the existence of a homogeneous solenoid of pseudo-arcs corresponding to each solenoid, but their uniqueness was left open. Hagopian and Rogers' classification did not assume that the solenoids of pseudo-arcs were necessarily obtainable by the construction of Rogers. There was thus the question of whether two solenoids of pseudo-arcs with decompositions to the same solenoid are necessarily homeomorphic. Furthermore, it is not readily evident that a solenoid of pseudo-arcs as defined in the classification of Hagopian and Rogers must necessarily be homogeneous. So if the answer to the uniqueness question is no, there is the question of whether every solenoid of pseudo-arcs is homogeneous.

Rogers [**17**] presented the above results at the spring topology conference at the University of Oklahoma in 1978 in a talk entitled "Almost everything you wanted to know about homogeneous, circle-like continua". He suggested that if someone gave an affirmative answer to the uniqueness question they could present the results under the title "Everything you wanted to know about homogeneous, circle-like continua". The author does not necessarily agree that such a result would provide everything one wanted to know about such continua. There remain questions about the mapping properties of such continua, the structure of their homeomorphism groups, and whether every homogeneous, atriodic (or tree-like, or hereditarily indecomposable) continuum is circle-like.

It is, however, the purpose of this paper to show that the solenoids of pseudo-arcs are unique and thus provide everything Jim Rogers wanted to know about homogeneous, circle-like continua. A solenoid (other than a simple closed curve) is locally homeomorphic to the product of a Cantor set with the unit interval. We will thus prove the uniqueness of the solenoids of pseudo-arcs in three steps. The case of the circle of pseudo-arcs has already been handled by Bing and Jones. We will first show that the Cantor set of pseudo-arcs is unique, then that the Cantor set of arcs of pseudo-arcs is unique, then that for a given solenoid the solenoid of pseudo-arcs is unique.

It is assumed that the reader is familiar with the basic results on chain maps and homeomorphisms of the pseudo-arc, and with the essence of the constructions by Bing and Jones involving arcs of pseudo-arcs. Our arguments and constructions will mimic these in many places. A *Cantor set of pseudo-arcs* will be a compact metric space with a continuous decomposition into pseudo-arcs such that the decomposition space is a Cantor set. We will show that such is homeomorphic to the product of a Cantor set with a pseudo-arc. A *Cantor set of arcs of pseudo-arcs* will be a one-dimensional compact metric space, each component of which is chainable, with a continuous decomposition into pseudo-arcs such that the decomposition space is homeomorphic to the product of the Cantor set with a unit interval. We will show that such is homeomorphic to the product of the Cantor set with an arc of pseudo-arcs. *A solenoid of pseudo-arcs* will be, as per Hagopian and Rogers, a circle-like continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a solenoid. We will restrict our attention to the case where the solenoid is not a simple closed curve.

We are now ready to prove the stated results.

THEOREM 1. *Every Cantor set of pseudo-arcs is homeomorphic to the product of a Cantor set with a pseudo-arc.*

PROOF. If C is a chain covering a Cantor set of pseudo-arcs M such that C essentially covers each maximal pseudo-arc in M, and $f_1, f_2, \ldots, f_n$ is a collection of chain maps onto C, then there exist chains $D_1, D_2, \ldots, D_n$ such that the inclusion of D_i in C agrees with the chain map f_i, $D_i^* \cap D_j^* = \phi$ if $i \neq j$, each D_i^* is clopen in M, and each D_i^* essentially covers each maximal pseudo-arc of M which intersects it. Thus if M is a Cantor set of pseudo-arcs and CP is the product of the Cantor set with a pseudo-arc, then one can obtain sequences of covers $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ of M and CP respectively such that:

(1) each of A_i and B_i is a finite union of disjoint chains with the union of each chain clopen in M and CP respectively;

(2) there is a one-to-one correspondence between the chains of A_i and those of B_i such that corresponding chains have the same number of links and follow the same patterns in the corresponding chains of A_{i-1} and B_{i-1} which contain them;

(3) A_{i+1} and B_{i+1} refine A_i and B_i respectively;

(4) mesh $A_i < 1/i$ and mesh $B_i < 1/i$;

(5) each maximal chain of A_i or B_i essentially covers each maximal pseudo-arc of M or CP, respectively, that intersects it.

The sequence of covers $\{A_i\}_{i+1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ induce a homeomorphism $h\colon M \to CP$. □

THEOREM 2. *Suppose P and Q are continuous Cantor sets of arcs of pseudo-arcs $\{p_{x,y}\}$, $\{q_{x,y}\}$ ($0 \leq x \leq 1$, $y \in C =$ Cantor set). Then each homeomorphism h that takes $\{p_{0,y}\} \cup \{p_{1,y}\}$ onto $\{q_{0,y}\} \cup \{q_{1,y}\}$ such that $h(p_{0,y} \cup p_{1,y}) = q_{0,z} \cup q_{1,z}$ with $h(p_{0,y}) = q_{0,z}$ and $h(p_{1,y}) = q_{1,z}$ for each $y \in C$, can be extended to a homeomorphism $\hat{h}$: $P \to Q$. The extension can be chosen such that if $h(p_{0,y}) = q_{0,z}$ then $\hat{h}(p_{x,y}) = q_{x,z}$ for each $0 \leq x \leq 1$ and $y \in C$.*

PROOF. This theorem parallels Theorem 10 of Bing and Jones [**4**], and a complete statement of its proof would parallel that given there. Rather than repeating the full details of that argument, we will outline the steps of the proof and indicate what modifications are necessary to obtain the result stated here.

We will get sequences of covers $\{D_i\}_{i=1}^{\infty}$ and $\{E_i\}_{i=1}^{\infty}$ of P and Q, respectively, to be used in defining the homeomorphism $\hat{h}$. Each of D_i and E_i will consist of a finite collection of disjoint chains, each with clopen union and such that if a component of P or Q intersects the union of one of these chains it is essentially covered by that chain. We will also obtain sequences of maps of covers, $\{H_i\}_{i=1}^{\infty}$ and $\{K_i\}_{i=1}^{\infty}$, each of which is a finite union of chain maps, with H_i taking D_i onto E_i and K_i taking E_{i+1} onto D_i.

The map H_i will be an approximation to the desired homeomorphism $\hat{h}$ in that for each point $p \in P$, there is an element d_i^j of D_i with $p \in d_i^j$ and $\hat{h}(p) \in H_i(d_i^j)$. The map K_i will be an approximation to $\hat{h}^{-1}$ in that for each point $q \in Q$ there is an element e_i^k of E_i with $q \in e_i^k$ and $\hat{h}^{-1}(q) \in K_i(e_i^k)$. Furthermore, K_i agrees with H_i^{-1} in that each element of E_{i+1} lies in its image under H_iK_i, and H_{i+1}^{-1} agrees with K_i in that each element of D_{i+1} lies in its image under K_iH_{i+1}. The D's, E's, H's, and K's will be defined in the order: E_1, D_1, H_1, E_2, K_1, D_2, H_2, E_3, $K_2, \ldots,$ with the meshes of E_i and D_i less than $1/2^i$.

The D's, E's, H's, and K's will be chosen such that for each $p \in p_{x,y}$ there is a decreasing sequence of elements $\{d_i^{n_i}\}$ of $\{D_i\}$ containing p such that $K_{i-1}H_i(d_{i+1}^{n_{i+1}}) = d_i^{n_i}$ and $\{H_i(d_i^{n_i})\}$ is a sequence of elements of $\{E_i\}$ whose closures contain a point $q \in q_{x,z}$. The homeomorphism $\hat{h}$ will be such that $\hat{h}(p) = q$. To define the D's, E's, H's, and K's properly, the homeomorphism $\hat{h}$ will be extended to certain elements of $\{p_{x,y}\}$ as we go.

$P(a, b)_y$ will denote the union of the elements $\{p_{x,y} \mid a \leq x \leq b\}$ and $P(a, b, M)$ will denote the union of the elements $\{p_{x,y} \mid a \leq x \leq b, y \in M\}$ where M is a clopen subset of the Cantor set; similarly for $Q(a, b)_y$ and $Q(a, b, M)$.

Now for the steps of the proof.

Step 1. In this step E_1 is defined and the homeomorphism $\hat{h}$ is extended to more elements of $\{p_{x,y}\}$.

(a) Consider a finite collection of disjoint chains E, as in the initial description of D_i and E_i, whose union irreducibly covers Q, each of mesh less than $\frac{1}{2}$.

(b) By the continuity of the collection $\{p_{x,y}\}$ there is a positive integer n such that if $0 \leq b - a \leq 1/n$ and M is a clopen subset of the Cantor set of diameter less than $1/n$, then the subchain of E irreducibly covering $Q(a, b)_x$ ($x \in M$) properly covers each element of $Q(a, b, M)$. If necessary each chain of E can be split lengthwise, so that its projection onto the Cantor set has diameter less than $1/n$. This collection of split chains is E_1.

(c) Extend the homeomorphism h already defined on $\{p_{0,y}\} \cup \{p_{1,y}\}$, $y \in C$ to $\{p_{i/n,y}\}$, $y \in C$. No care in extending is necessary at this stage other than that $\hat{h}$ takes $p_{i/n,y}$ homeomorphically onto $q_{i/n,z}$ and is a homeomorphism on each of the Cantor sets of pseudo-arcs $p_{i/n} \times C$.

Step 2. In this step D_1 and H_1 are defined and the map $\hat{h}$ is extended further by defining it on more elements of $\{p_{x,y}\}$. The following conditions are to be satisfied.

(I) If element d_j of D_1 intersects $p_{i/n,y}$ then $H_1(d_j)$ contains $\hat{h}(d_j \cap p_{i/n,y})$.

(II) For each x, H_1 takes any subchain of D_1 that properly covers $p_{x,y}$ onto a subchain of E_1 that properly covers $q_{x,z}$.

This step can be divided into parts (a)–(i) paralleling the parts in the proof of Bing and Jones [**4**]. Where chains of small mesh are chosen by them so that sufficiently near pseudo-arcs are covered by subchains with the same mapping properties, in our case these chains are, if necessary, split lengthwise (still with clopen projections into C) so that nearby pseudo-arcs in different components of P have similar covering properties by the subchains under consideration. When an ε is chosen it is sufficiently small to satisfy the desired properties for pseudo-arcs in nearby components as well as in a given component.

In part (i) of this step, k is chosen sufficiently large to satisfy the desired property for each $p_{x,y}$, $a \leq x \leq b$, $y \in M$, where $0 \leq b - a \leq 1/k$ and M is a clopen subset of C of diameter less than or equal to $1/k$. $\hat{h}$ is then extended to all of the $p_{i/k,y}$ such that $h(p_{i/k,y}) = q_{i/k,z}$, and for each element d_j of the subchain of D_1 properly covering $p_{i/k,y}$, $h(p_{i/k,y} \cap d_j) \subset H_1(d_j)$. Also $\hat{h}$ is a homeomorphism on each $p_{i/k} \times C$.

Step 3. A collection of chains E_2 covering Q and a collection of chain maps K_2 of E_2 onto the chains of D_1 is defined, and the map $\hat{h}$ is extended to additional elements of $\{p_{x,y}\}$. This time E_2 is a collection of $\frac{1}{4}$ chains whose union irreducibly covers Q, and K_1 is a collection of chains maps of E_2 onto chains of D_1 such that:

(I) If link e_i of E_2 intersects $q_{j/k,z}$, then $h^{-1}(e_i \cap q_{j/k,z}) \subset K_1(e_i)$.

(II) For each x, z, K_1 takes any subchain of E_2 properly covering $q_{x,z}$ onto a subchain of D_1 properly covering $p_{x,y}$.

(III) $e_i \subset H_1 K_1(e_i)$ for each $e_i \in E_2$.

Again this step parallels step 3 of Bing and Jones [**4**] with adjustments being made in ε to take into account nearby pseudo-arcs in distinct components, and chains being split lengthwise as needed for this same reason, as described above.

Steps $4, 5, \ldots$, are essentially repetitions of Step 3, with the modifications indicated above, alternating between describing D_i and H_i and defining E_i and K_i. As described initially, the covers $\{D_i\}$, $\{E_i\}$ and maps $\{H_i\}$, $\{K_i\}$ induce a homeomorphism $\hat{h}$: $P \to Q$ satisfying the desired conditions and agreeing with each of the partial definitions of $\hat{h}$ made during the construction.

It might be observed that $\hat{h}$ can, if desired, be constructed to be a product in the middle of the Cantor set of arcs of pseudo-arcs, fanning out toward the ends to agree with the initial end homeomorphisms on individual components. □

THEOREM 3. *For a given solenoid S, there is a unique solenoid of pseudo-arcs P with S as decomposition space.*

PROOF. A solenoid can be obtained from two sets of Cantor sets of arcs by identifying endpoints, the particular identification determining which solenoid, if any, is obtained. For a solenoid of pseudo-arcs there are two corresponding sets of arcs of pseudo-arcs with identifications. Theorem 1 shows that for any two solenoids of pseudo-arcs one can obtain homeomorphisms between the end Cantor sets of pseudo-arcs of the Cantor sets of arcs of pseudo-arcs making up the two continua. By Theorem 2, these homeomorphisms between the ends can be extended to homeomorphisms of the two pairs of Cantor sets of arcs of pseudo-arcs as long as components match up properly. The matching of components is determined by the identifications giving the respective solenoids. □

As mentioned earlier, this result now completes the classification of homogeneous, circle-like continua. With previous results it gives a listing of all such continua up to equivalence by homeomorphisms. It is still of interest though to determine whether this class is larger than it might appear on the surface. For example, all known nondegenerate homogeneous continua which are either planar, atriodic, tree-like, hereditarily decomposable, or hereditarily indecomposable are also cirlce-like. Must this actually be true of any nondegenerate homogeneous member of any of the above five classes of continua?

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