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A NOTE ON COMPACT GROUPS

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ABSTRACT. We show that the product of certain subsets in a compact connected topological group is the group itself.

Let G be a connected topological group. It is well known that G is generated by any neighborhood V of the identity 1 of G, i.e. $G = \bigcup_{n=1}^{\infty} V^n$. If G is also compact, then $G = V^k$ for some k. It is natural to ask whether the above statement is true if we replace the neighborhood of 1 by some other types of subsets of G. The purpose of this note is to show one such possibility. Precisely, we prove:

PROPOSITION. *Let G be a compact connected Hausdorff topological group and μ the (normalized) Haar measure on G. Let $A_1, A_2, \ldots$ be a sequence of Borel measurable sets in G such that* $\inf \mu(A_i) > 0$. *Then $G = A_1 A_2 \cdots A_n$ for some n.*

First, we need

LEMMA. *Let A and B be Borel subsets in G with positive measure. Then AB has nonvoid interior.*

PROOF. This is a special case of a well-known general result. For a proof, see [**3**, (20.17), p. 296].

PROOF (OF THE PROPOSITION). We may suppose by the Lemma that A_1 is an open subset. There are elements $a_1, a_2, \ldots$ in G such that $1 \in A_1 a_1$, and $1 \in a_{n-1}^{-1} A_n a_n$, for $n > 1$. Let $B_1 = A_1 a_1$, $B_n = a_{n-1}^{-1} A_n a_n$. Then $\mu(B_n) = \mu(A_n)$. Since $A_1 A_2 \cdots A_n = B_1 B_2 \cdots B_n a_n^{-1}$, it suffices to prove the Proposition for the sequence $B_1, B_2, \ldots$ of Borel sets containing the identity element.

Let $S = \{x \in G \mid x \in B_i$ for infinitely many indices$\}$. Then $S = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} B_i$. We note that $1 \in S$ and $\mu(S) \geqslant \inf \mu(B_i)$. Let S^* be the semigroup generated by S, i.e. $S^* = \bigcup_{n=1}^{\infty} S^n$. Then $1 \in S \subset S^*$. Because S has positive measure, by the Lemma we know that S^* has nonvoid interior W. Now we shall prove that 1 is in the interior W of S^*. Let x be any element in W. Since G is compact, the closure of the semigroup generated by x^{-1} is a compact subgroup; in particular, it contains 1. Thus x^{-n} is in the neighborhood $x^{-1}W$ of 1 for some $n \geqslant 1$. Then $x^{1-n} \in W$, and $1 = x^{n-1} \cdot x^{1-n} \in x^{n-1}W \subset W^n = W$. In other words, S^* is a neighborhood of 1. Since G is connected we conclude that $G = S^*$ (cf. the beginning of this note). Finally, since S^* is generated by the elements which appear infinitely often in the

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sequence $B_1, B_2, \ldots, S^* \subseteq \bigcup_{n=1}^{\infty} B_1 B_2 \cdots B_n$. Using the fact that $G = S^*$ is compact and each $B_1 B_2 \cdots B_n$ is open, we conclude that $G = S^* = B_1 B_2 \cdots B_k$ for some k, and the proof is complete.

REMARK. Observe that, given a sequence of open neighborhoods $V_1 V_2, \ldots$ of 1, it is not always true that $G = V_1 V_2 \cdots V_k$ for some k. Simple examples such as taking small intervals in the circle illustrate this fact.

For earlier works in this direction, but on abelian locally compact groups, we refer to [**1, 2, 5**] and references therein. This note shows certain kinds of ergodicity of subsets in compact groups. For applications of this kind. we refer to [**4**].

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