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CHARACTERIZATION OF QUASI-UNITS IN TERMS OF EQUILIBRIUM POTENTIALS[1]

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ABSTRACT. In the cone of nonnegative superharmonic functions on a bounded euclidean region Ω, quasi-units were introduced as those elements invariant under the infinitesimal generator of the fundamental one-parameter semigroup of operators S_λ ($\lambda \geq 0$). All harmonic measures and capacitary potentials are quasi-units, but the latter class has more extensive closure properties. Quasi-units arise naturally under various operations of classical potential theory and have important applications, for example in proving that the convex set of Green's potentials u of positive mass distributions on Ω with $u \leq 1$ has as its extreme points precisely the capacitary potentials. Some new properties of quasi-units are developed here. In particular, it is shown that quasi-units can be characterized as limits of increasing sequences of continuous equilibrium potentials for which the equilibrium values tend to 1.

1. Let $\mathfrak{A}$ be the set of all nonnegative superharmonic functions on a bounded region Ω of euclidean space. In $\mathfrak{A}$ there is a fundamental class of functions which contains all harmonic measures and capacitary potentials but has more extensive closure properties than the union of those two classes. Elements of this fundamental class are examples of what the present authors have termed "quasi-units". Although quasi-units were introduced and studied in a much more abstract setting (where they appear as generalizations of the Freudenthal-Yosida quasi-units in the theory of vector lattices), it should be emphasized that the present discussion is self-contained and assumes only a knowledge of classical potential theory.

Our purpose here is to present some new properties and characterizations of quasi-units in $\mathfrak{A}$. It is shown, in particular, that these quasi-units can be characterized as limits of increasing sequences of equilibrium potentials for which the equilibrium values tend to 1. Thus, although the limit functions themselves are not, in general, equilibrium potentials (or harmonic measures or other familiar objects of classical potential theory), they must always be quasi-units. This indicates why a theory of quasi-units is needed even in classical potential theory. There are other reasons as well, and we shall take a broader look at the role of quasi-units at the end of this section.

A definition of quasi-units can be framed in terms of certain operators $S_{\lambda e}$ ($\lambda \geq 0$) on $\mathfrak{A}$. Here, and throughout the discussion, e will be fixed as any nonzero element of $\mathfrak{A}$. In the most classical situations e is taken as the constant function $e = 1$, and it

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will be convenient in the general case to regard e as a "unit element". Relative to this element, the operators S_{λ_e} are defined by setting

$$S_{\lambda_e}u = \hat{R}(u - \lambda e)^+ \qquad (u \in \mathfrak{U}), \tag{1.1}$$

where $\hat{R}$ denotes the regularized reduced function operator. That is, for ϕ any nonnegative function on Ω, $\hat{R}\phi$ is the unique function in $\mathfrak{U}$ which coincides almost everywhere with the reduced function $R\phi = \inf\{u \in \mathfrak{U}: u \geqslant \phi\}$. We now define an *e-quasi-unit* as any element u of $\mathfrak{U}$ such that the equality

$$S_{\lambda_e}u = (1 - \lambda)u \tag{1.2}$$

holds for all λ on the interval (0.1). It is easy to see that all e-quasi-units u satisfy $u \leqslant e$ and that (1.2) is actually equivalent to the pair of conditions

$$u \leqslant e \quad \text{and} \quad S_{\lambda_e}u \geqslant (1 - \lambda)u. \tag{1.3}$$

It will be recalled that, for K a compact subset of the region Ω and λ a positive constant, the *equilibrium potential* for the set K with equilibrium value λ is just $\lambda \hat{R}_1^K$. Here we use $\hat{R}_\phi^E$ in the usual way to denote $\hat{R}(\phi\chi_E)$, where χ_E is the characteristic function of E. It is readily verified that equilibrium potentials, defined in this way, are necessarily Green's potentials over Ω. In what follows, we shall have use for a natural extension of the concept of an equilibrium potential, in which the constant function 1 is replaced by our generalized unit element e, and the compact set K is replaced by an arbitrary subset E of Ω. We shall refer to $\lambda \hat{R}_e^E$ as a *generalized equilibrium function* for the set E, having *equilibrium value* λ (relative to e).

Our main result can now be stated, in the classical case of $e = 1$, as follows: a function u on Ω is a 1-quasi-unit if and only if it is the limit of an increasing sequence of continuous equilibrium potentials with equilibrium values tending to 1. More generally, a corresponding characterization holds for e-quasi-units, namely

THEOREM 1.1. *Let e be any continuous nonzero function in $\mathfrak{U}$. Then a function u on Ω is an e-quasi-unit if and only if it is the limit of an increasing sequence $\{\lambda_n \hat{R}_e^{K_n}\}$ of continuous generalized equilibrium potentials with all K_n compact and $\lambda_n \to 1$.*

Before we turn to the proof of Theorem 1.1 and related results, it may be of interest to have some background information on the development of the quasi-unit concept and the scope of the resulting theory. The notion of a quasi-unit was introduced in **[2]** in the abstract setting of a strongly superharmonic cone. There it was shown that the operators S_{λ_e} form a one-parameter semigroup, and quasi-units were defined as those elements invariant under the infinitesimal generator of the semigroup. Our defining property (1.2) appears in **[2]** as one of the characterizations of quasi-units, and it is also equivalent to the defining property adopted in the algebraic setting of **[3]**. As noted in **[2]** (see also **[5]**), property (1.2) will hold for all λ on (0, 1) if it holds for one λ on that interval. A large number of closure properties of quasi-units are established in **[2]**–**[5]**, along with various characterizations and other properties. Quasi-units are applied in **[2]** to arrive at an extension of the classical Freudenthal spectral theorem, and further applications can be found in **[5]**, where quasi-units are used to obtain characterizations of the extreme points of certain convex sets in $\mathfrak{U}$. In particular, it is shown in [5] that the convex set of Green's potentials in $\mathfrak{U}$ which are bounded above by 1 has as its extreme points precisely the capacitary potentials in $\mathfrak{U}$. We note also that [5] treats the theory of quasi-units in the classical and axiomatic settings independently of the algebraic theory in [2] and [3] and makes use of the theory of finitely harmonic functions (as developed by B. Fuglede [7]) to obtain new results in those settings. It should be remarked that a discussion of the quasi-units of K. Yosida can be found in his book [8].

2. Some properties and characterizations of quasi-units. We proceed to derive some new properties and characterizations of quasi-units that hold in the present setting. Throughout the present section $\mathfrak{U}$ can, in fact, be taken as the set of all nonnegative superharmonic functions on an arbitrary harmonic space, and Theorem 2.3 even remains valid in the abstract setting of [2].

For u an e-quasi-unit and $0 < \lambda < 1$, it is easy to see that

$$\lambda u \leqslant \lambda \hat{R}_e^{[u \geqslant \lambda e]} \leqslant u \tag{2.1}$$

and, similarly,

$$\lambda u \leqslant \lambda \hat{R}_e^{[u > \lambda e]} \leqslant u. \tag{2.2}$$

For example, to prove (2.1), one has only to observe that

$$(u - \lambda e)^+ \leqslant (1 - \lambda) e \chi_{[u \geqslant \lambda e]}$$

and, hence,

$$(1 - \lambda) u = S_{\lambda e} u \leqslant (1 - \lambda) \hat{R}_e^{[u \geqslant \lambda e]}.$$

An application of (2.1) yields the following characterization of e-quasi-units in terms of regularized reduced functions.

THEOREM 2.1. *A function u in $\mathfrak{U}$ is an e-quasi-unit if and only if it satisfies $u \leqslant e$ and*

$$\lim_{\lambda \to 1-} \hat{R}_e^{[u \geqslant \lambda e]} = u. \tag{2.3}$$

PROOF. Obviously, if u is an e-quasi-unit, then $u \leqslant e$ and (2.3) follows from (2.1). For the converse we draw on (1.3) and show that (2.3) implies $S_{\lambda e} u \geqslant (1 - \lambda) u$ for $0 < \lambda < 1$. Thus, assuming (2.3), we fix λ and λ' with $0 < \lambda < \lambda' < 1$ and note that the evident inequality $(u - \lambda e)^+ \geqslant (\lambda' - \lambda) e \chi_{[u \geqslant \lambda' e]}$ results in

$$S_{\lambda e} u \geqslant (\lambda' - \lambda) \hat{R}_e^{[u \geqslant \lambda' e]}.$$

A passage to the limit as $\lambda' \to 1$ completes the proof.

The same argument applies with the set $[u \geqslant \lambda e]$ replaced by $[u > \lambda e]$, so that (2.3) can be replaced by

$$\lim_{\lambda \to 1-} \hat{R}_e^{[u > \lambda e]} = u. \tag{2.4}$$

Also, Theorem 2.1 leads to another characterization of e-quasi-units in terms of regularized reduced functions. Here it will be convenient to employ the operators $B_{\lambda e}$ introduced in [1] (see p. 278 of [1], where the notation B_λ is used, and Theorem

4.3 of [3], which applies these operators in characterizing e-singular elements). The operators $B_{\lambda e}$ are defined by

$$B_{\lambda e}u = \hat{R}_u^{[u \geq \lambda e]} \qquad (u \in \mathcal{U}). \tag{2.5}$$

We show that the e-quasi-units in $\mathcal{U}$ are precisely those elements $u \leq e$ which are invariant under the operators $B_{\lambda e}$ with $0 < \lambda < 1$.

THEOREM 2.2. *A function u in $\mathcal{U}$ is an e-quasi-unit if and only if it satisfies $u \leq e$ and $B_{\lambda e}u = u$, i.e.,*

$$\hat{R}_u^{[u \geq \lambda e]} = u, \tag{2.6}$$

for all λ with $0 < \lambda < 1$.

PROOF. To see that $B_{\lambda e}u = u$ must hold for $0 < \lambda < 1$ when u is an e-quasi-unit, we first observe that $B_{\lambda e}u \geq \lambda \hat{R}_e^{[u \geq \lambda e]}$. Hence, by Theorem 2.1,

$$\lim_{\lambda \to 1-} B_{\lambda e}u \geq u.$$

Since $B_{\lambda e}u$ is plainly a decreasing function of λ, we conclude that $B_{\lambda e}u \geq u$ holds for all λ on $(0, 1)$. Our assertion then follows from the fact that the reverse inequality always holds. For the converse, suppose that $u \leq e$ and $B_{\lambda e}u = u$ for λ on $(0, 1)$. Then, letting $\lambda \to 1-$ in the inequalities

$$u/\lambda \geq \hat{R}_e^{[u \geq \lambda e]} \geq \hat{R}_u^{[u \geq \lambda e]} = B_{\lambda e}u = u$$

yields (2.3), completing the proof.

Here, as before, we observe that (2.6) can just as well be replaced by the condition

$$\hat{R}_u^{[u > \lambda e]} = u. \tag{2.7}$$

We next establish a modified monotone convergence property of quasi-units. This makes use of the elementary fact that, for u and u_n in $\mathcal{U}$ $(n = 1, 2, \ldots)$,

$$u_n \uparrow u \Rightarrow S_{\lambda e}u_n \uparrow S_{\lambda e}u. \tag{2.8}$$

THEOREM 2.3. *Let $\{u_n\}$ be a sequence of e-quasi-units and $\{\lambda_n\}$ a sequence of positive numbers with $\lambda_n \to 1$. If $\{\lambda_n u_n\}$ is an increasing sequence having limit u, then u is an e-quasi-unit.*

PROOF. Fixing λ arbitrarily on the interval $(0, 1)$, we assume without loss of generality that $0 < \lambda < \lambda_n$ for all n. The desired condition (1.2) then follows at once by passing to the limit on n in the equality

$$S_{\lambda e}(\lambda_n u_n) = \lambda_n S_{(\lambda/\lambda_n)e}u_n = (\lambda_n - \lambda)u_n.$$

It has been remarked earlier that all capacitary potentials are quasi-units. In fact, all regularized reduced functions $u = \hat{R}_e^E$ $(E \subset \Omega)$ are e-quasi-units. Although this appears already as Theorem 6.6 of [5], the proof is so simple that we include it here for the sake of completeness. Let λ be any point of $(0, 1)$, and note that u satisfies the inequality $u \leq S_{\lambda e}u + \lambda e$. The fact that $u = e$ quasi everywhere on E yields $e \leq S_{\lambda e}u + \lambda e$ quasi everywhere on E, so that $u = \hat{R}_e^E \leq (1/(1 - \lambda))S_{\lambda e}u$. Since, obviously,

$u \leqslant e$, conditions (1.3) are satisfied, and the proof is complete. Taking the u_n in Theorem 2.3 as regularized reduced functions results in

COROLLARY 2.4. *Let $\{\lambda_n \hat{R}_e^{E_n}\}$ be an increasing sequence of generalized equilibrium functions with equilibrium values λ_n tending to* 1. *Then the limit u of the sequence is an e-quasi-unit.*

We proceed to show that e-quasi-units can actually be characterized as limits of the sort considered in Corollary 2.4. The key to the proof lies in the following bracketing property: if u is an e-quasi-unit and λ_1, λ_2 positive numbers with $\lambda_1 < \lambda_2$, then

$$\lambda_1 u \leqslant \lambda_1 \hat{R}_e^{[u > (\lambda_1/\lambda_2)e]} \leqslant \lambda_2 u. \tag{2.9}$$

These inequalities have a remarkably simple derivation; one has only to go back to (2.1) and put $\lambda = \lambda_1/\lambda_2$.

THEOREM 2.5. *A function u on Ω is an e-quasi-unit if and only if it is the limit of an increasing sequence $\{\lambda_n \hat{R}_e^{E_n}\}$ of generalized equilibrium functions with equilibrium values λ_n tending to* 1. *Moreover, when e is continuous, the sets E_n can be taken as open.*

PROOF. In view of Corollary 2.4, there remains only to show that every e-quasi-unit u can be expressed as such a limit. That this is the case is immediate from (2.9) by taking $\{\lambda_n\}$ as any strictly increasing sequence of positive numbers converging to 1 and $\{E_n\}$ as the sequence of sets $E_n = [u > (\lambda_n/\lambda_{n+1})e]$.

3. Proof of Theorem 1.1. Here e is continuous. We start with the sequence $\{\lambda_n \hat{R}_e^{E_n}\}$ of Theorem 2.5 and exhaust each of the open sets E_n by an increasing sequence of open sets A_{nk} $(k = 1, 2, \ldots)$ with each $\bar{A}_{nk}$ a compact subset of $A_{n(k+1)}$. It will be required further that the sets A_{nk} are finite unions of spheres, so that all their boundary points are regular. The regularized reduced functions $\hat{R}_e^{\bar{A}_{nk}}$ are then just the functions equal to e on $\bar{A}_{nk}$ and equal, on the complement of $\bar{A}_{nk}$, to the solution of the Dirichlet problem for the boundary values given by e on $\bar{A}_{nk}$ and 0 on $\partial\Omega$. Consequently, these functions are continuous Green's potentials on Ω which assume the values given by e on $\bar{A}_{nk}$. In this sense they appear as generalized capacitary potentials (relative to e).

Now let $\lambda_{nk} = \lambda_n - 1/k$, and consider the generalized equilibrium potentials

$$\lambda_{nk} \hat{R}^{\bar{A}_{nk}} \qquad (n, k = 1, 2, \ldots). \tag{3.1}$$

For fixed n the functions in (3.1) form a sequence over the index k which converges upward to the function $\lambda_n \hat{R}_e^{E_n}$, and we plainly have

$$\lambda_{nk} \hat{R}_e^{\bar{A}_{nk}} < \lambda_n \hat{R}_e^{E_n}. \tag{3.2}$$

(For the convergence assertion see, for example, C. Constantinescu and A. Cornea [**6**, Corollary 4.2.2, p. 114].)

The next step is to enumerate the functions (3.1) as a single sequence $\{u_j\}$. Thus, each of the functions u_j $(j = 1, 2, \ldots)$ is a continuous generalized equilibrium potential for some corresponding compact set K_j, with u_j taking on values $\mu_j e$ on K_j for some $\mu_j \in (0, 1)$. It is readily verified that the sequence $\{u_j\}$ is filtering upward to u. Indeed, we obviously have $u = \sup_j u_j$, so that there remains only to show that for any two functions u_p and u_q in the sequence there is a third function u_r dominating both of them, i.e.,

$$\mu_p \hat{R}_e^{K_p} \leqslant \mu_r \hat{R}_e^{K_r} \quad \text{and} \quad \mu_q \hat{R}_e^{K_q} \leqslant \mu_r \hat{R}_e^{K_r}. \tag{3.3}$$

To prove this, we first choose an index n such that

$$\mu_p \hat{R}_e^{K_p} < \lambda_n \hat{R}_e^{E_n} \quad \text{and} \quad \mu_q \hat{R}_e^{K_q} < \lambda_n \hat{R}_e^{E_n}. \tag{3.4}$$

We then appeal to the following version of Dini's theorem: if $\{f_k\}$ is an increasing sequence of continuous functions on a compact set K and f a continuous function on K such that $f < \lim f_k$, then $f < f_k$ for all sufficiently large indices k. By taking the functions f_k as in (3.1) with n fixed according to (3.4), it is clear that the existence of a function u_r for which (3.3) holds is a direct consequence of (3.4). The theorem follows.

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