

## REALIZABILITY AND NONREALIZABILITY OF DICKSON ALGEBRAS AS COHOMOLOGY RINGS

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**ABSTRACT.** Fix a prime  $p$  and let  $V$  be an  $n$ -dimensional vector space over  $\mathbf{Z}/p$ . The general linear group  $GL(V)$  of  $V$  acts on the polynomial ring  $P(V)$  on  $V$ . The ring of invariants  $P(V)^{GL(V)}$  has been computed by Dickson, and we denote it by  $D^*(n)$ . If we grade  $P(V)$  by assigning the elements of  $V$  the degree 2, then  $D^*(n)$  becomes a graded polynomial algebra on generators  $Y_1, \dots, Y_n$  of degrees  $2p^n - 2p^{n-1}, \dots, 2p^n - 2$ . The mod  $p$  Steenrod algebra acts on  $P(V)$  in a unique way compatible with the unstability condition and the Cartan formula. The  $GL(V)$  action commutes with the Steenrod algebra action, and so  $D^*(n)$  inherits the structure of an unstable polynomial algebra over the Steenrod algebra. In this note we determine explicit formulas for the action of the Steenrod algebra on the polynomial generators of  $D^*(n)$ . As a consequence we are able to decide exactly which Dickson algebras can be  $\mathbf{Z}/p$  cohomology rings.

**0. Introduction.** Let  $p$  denote a prime integer and let  $V$  be an  $n$ -dimensional vector space over the Galois field  $\mathbf{Z}/p$ . The general linear group  $GL(V)$  of  $V$  acts on  $P(V)$ , the polynomial ring over  $V$ . The ring of invariants  $P(V)^{GL(V)}$  will be denoted by  $D^*(n)$ . We regard  $P(V)$  as being graded by requiring that the elements of  $V$  have grading degree 2. Then  $D^*(n)$  is a graded subalgebra of  $P(V)$ . The graded polynomial algebra  $P(V)$  carries in a unique way the structure of an unstable algebra over the Steenrod algebra (see for example [15, 13 or 2] for the definition of the relevant terms) determined by

$$P^k v = \begin{cases} v: & k = 0, \\ v^p: & k = 1, \\ 0: & k > 1, \end{cases} \quad \forall v \in V, p > 2,$$

$$\beta v = 0,$$

and

$$Sq^k v = \begin{cases} v: & k = 0, \\ v^2: & k = 2, \\ 0: & \text{otherwise,} \end{cases} \quad p = 2 \quad \text{and} \quad \forall v \in V.$$

The action of  $GL(V)$  is compatible with the action of the Steenrod algebra and thus  $D^*(n)$  inherits the structure of an unstable algebra over the mod  $p$  Steenrod algebra. The algebra structure of  $D^*(n)$  has been determined by Dickson [4] (see also (1.5) below) who showed

$$D^*(n) \simeq \mathbf{Z}/p[Y_1, \dots, Y_n]: \quad \deg Y_i = 2p^n - 2p^{n-i}; \quad p \text{ odd}; \quad i = 1, \dots, n.$$

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The Dickson algebra plays a central role in determining which algebras over the Steenrod algebra occur as rings of invariants (see e.g. [2] where this is well disguised or [14] where it is brought to the fore). For large primes this in turn shows that the polynomial algebras that occur as cohomology rings are exactly those constructed in [3]. It is therefore quite natural to ask if the Dickson algebra itself can occur as a cohomology ring. Our main result settles this question for all primes. Specifically we show

**THEOREM.** *Let  $p$  be a prime and  $n$  a positive integer. Then there exists a topological space  $X(n)$  such that*

$$H^*(X(n); \mathbf{Z}/p) \simeq D^*(n)$$

*as algebras over the Steenrod algebra iff  $n = 1$ , or  $n = 2$  and  $p \leq 3$ .*

For  $n = 1$ ,  $D^*(1) \simeq \mathbf{Z}/p[Y]$  where  $\deg Y = 2p - 2$ , and a relevant space  $X(1)$  realizing  $D^*(1)$  is a delooping of  $S^{2p-3}(p)$  which has been constructed by Holzager [8] and Sullivan [17]. A space with cohomology mod 3 isomorphic to  $D^*(2) \simeq \mathbf{Z}/3[Y_1, Y_2]$ ,  $\deg Y_1 = 12$ ,  $\deg Y_2 = 16$ , has recently been constructed by Zabrodsky [18], utilizing Friedlander's exceptional isogeny [6]  $\theta: BF_4(\frac{1}{2}) \rightarrow BF_4(\frac{1}{2})$ . This corrects an oversight in [13] that was noted in [5]. (In [5] however the type  $(8, 12) \bmod 3$  is incorrectly eliminated as a cohomology ring. The type in question occurs as for example the ring of invariants  $P(V)^{SL(V)}$  where  $V$  is a 2-dimensional vector space over  $\mathbf{Z}/3$ . However more detailed calculation shows this latter type is not a cohomology ring.) For  $p = 2$ ,  $D^*(1)$  and  $D^*(2)$  are the  $\mathbf{Z}/2$  cohomology of  $CP(\infty)$  and  $BSU(3)$  respectively.

The organization of this paper is as follows. In §1 we determine the structure of  $D^*(n)$  as an unstable algebra over the Steenrod algebra. In the process we rederive the theorem of Dickson in the spirit of Adams and Wilkerson [2, §§3 and 5]. Combining this with elementary computations with secondary operations we arrive at the nonrealization theorem in §2. In §3 we discuss the (really odd !) prime 2.

For further connections between the Dickson algebra and algebraic topology we refer the reader to [15]. We wish to thank Clarence Wilkerson for a useful correspondence concerning the case  $p = 2$ .

**1. The Steenrod algebra structure of the Dickson algebra.** To simplify the exposition in this section we will suppose that  $p$  is an odd prime. The modifications for the prime 2 are left to the reader. The results are at any rate formulated in §2.

**NOTATIONS AND DEFINITIONS.** We denote by  $\mathfrak{P}^*$  the algebra of Steenrod reduced powers (no Bockstein). The dual Hopf algebra  $\mathfrak{P}_*$  has been determined by Milnor [11] who showed

$$\mathfrak{P}_* \simeq \mathbf{Z}/p[\xi_1, \dots, \xi_n, \dots]: \quad \deg \xi_i = 2p^i - 2; \quad i > 0.$$

Thus a basis for the primitive elements of  $\mathfrak{P}^*$  is given by the elements  $P^{\Delta_i} \in \mathfrak{P}^{2p^i-2}$  dual to  $\xi_i$  in the usual monomial basis of  $\mathfrak{P}_*$ . The elements  $P^{\Delta_i}$  can be inductively defined by

$$P^{\Delta_1} := P^1, \quad P^{\Delta_{i+1}} := [P^{p^i}, P^{\Delta_i}].$$

By an unstable algebra  $A^*$  over the Steenrod algebra we understand an evenly graded algebra over the Hopf algebra  $\mathfrak{S}^*$  in the usual sense [12] that satisfies in addition the unstability conditions

$$P^k(a) = \begin{cases} a^p: & 2k = \text{deg } a, \\ 0: & 2k > \text{deg } a. \end{cases}$$

The category of unstable algebras over the Steenrod algebra is denoted by  $UnAl/\mathfrak{S}^*$ .

Being primitive, the elements  $P^{\Delta_i}$  act as derivations on any  $A^* \in UnAl/\mathfrak{S}^*$ . It is convenient to introduce a further derivation  $P^{\Delta_0}$  of degree zero by the rule

$$P^{\Delta_0}(a) = da: \quad \forall a \in A^{2d}.$$

(N.B.  $P^{\Delta_0} \notin \mathfrak{S}^*$ .) The following result of Adams and Wilkerson [2, Theorem 5.1] is fundamental.

**PROPOSITION 1.1.** *Let  $A^* \in UnAl/\mathfrak{S}^*$  and suppose that there is an upper bound on the number of algebraically independent elements in  $A^*$ . Then there exists an integer  $n \geq 0$  such that any  $n + 1$  derivations  $P^{\Delta_0}, \dots, P^{\Delta_n}$  are linearly dependent, but any  $n$  distinct derivations  $P^{\Delta_i}$  are linearly independent.*

Linear dependence and independence as used in (1.1) should be understood as referring to the left  $A^*$  module structure of  $\text{End}(A^*)$ . An essential point in the proof of (1.1) is the following derivation lemma, which we will presently need ourselves.

**DERIVATION LEMMA.** *Let  $k$  be a perfect field and  $A^*$  a graded  $k$  algebra. Suppose we are given derivations*

$$\partial_1, \dots, \partial_n: \quad A^* \rightarrow A^*,$$

and elements  $x_1, \dots, x_n \in A^*$  such that

$$\text{Det}(\partial_i x_j) \neq 0.$$

Then  $x_1, \dots, x_n \in A^*$  are algebraically independent.

For a proof see [19, pp. 126–127].

**DEFINITION.** Let  $V$  be an  $n$ -dimensional vector space over  $\mathbf{Z}/p$ . Set  $S^* := P(V)$  and introduce the polynomial

$$f(X) := \prod_{v \in V} (X - v) \in S^*[X]$$

where  $X$  is an indeterminate of degree 2.

Note that the polynomial  $f(X)$  is invariant under the action of  $\text{GL}(V)$  on  $S^*[X]$ , so the coefficients belong to  $D^*(n)$ , so there are elements  $Y_1, \dots, Y_n \in D^*(n)$  such that

$$f(X) = X^{p^n} + Y_1 X^{p^n - 1} + \dots + Y_n X.$$

In addition the roots of  $f(X)$  form the vector space  $V$ .

The derivation lemma shows that the derivations  $P^{\Delta_0}, \dots, P^{\Delta_n}$  are linearly dependent over  $S^*$ . Thus we may choose  $d_0, \dots, d_n \in S^*$  such that

$$d_n P^{\Delta_0} + \dots + d_0 P^{\Delta_n} = 0 \in \text{End}(S^*).$$

Having made such a choice introduce the polynomial

$$\Delta(X) := d_n X + d_{n-1} X^p + \dots + d_0 X^{p^n} \in S^*[X].$$

LEMMA 1.2. *With the notations preceding  $\Delta(X) = d_0 f(X)$ .*

PROOF. The action of the derivations  $P^{\Delta_i}$  on 2-dimensional classes can be computed inductively from the instability condition and the formula  $P^{\Delta_{i+1}} = [P^{p^i}, P^{\Delta_i}]$ . The result is

$$P^{\Delta_i} v = v^{p^i} \quad i = 0, 1, \dots, \text{deg } v = 2.$$

Thus for  $v \in V$  we have

$$0 = (d_n P^{\Delta_0} + \dots + d_0 P^{\Delta_n})(v) = d_n v + d_{n-1} v^p + \dots + d_0 v^{p^n} = \Delta(v).$$

That is the elements of  $V$  are roots of the polynomial  $\Delta(X)$ . From the definition of  $f(X)$  it follows that  $f(X)$  divides  $\Delta(X)$  in  $S^*[X]$ . But  $\Delta(X)$  and  $f(X)$  both have degree  $p^n$  in  $X$ , so  $\Delta(X) = d_0 f(X)$  as claimed.  $\square$

LEMMA 1.3. *Suppose  $A^* \in \text{UnAl}/\mathcal{P}^*$  and*

$$(*) \quad h_0 P^{\Delta_0} = h_1 P^{\Delta_{r_1}} + \dots + h_n P^{\Delta_{r_n}} \in \text{End}(A),$$

where  $h_0$  is a  $p$ th power. Then

$$h_0 P^{\Delta_r} = (P^{\Delta_r} h_1) P^{\Delta_{r_1}} + \dots + (P^{\Delta_r} h_n) P^{\Delta_{r_n}}$$

for all  $r \geq 0$ .

PROOF. First of all recall the commutation relation

$$[P^{\Delta_i}, P^{\Delta_j}] = \begin{cases} 0 & j \neq 0, \\ P^{\Delta_i} & j = 0. \end{cases}$$

Let  $x \in A^*$  be arbitrary. From (\*) we get

$$(**) \quad h_0 P^{\Delta_0}(x) = h_1 P^{\Delta_{r_1}}(x) + \dots + h_n P^{\Delta_{r_n}}(x).$$

Apply  $P^{\Delta_r}$  to both sides of (\*\*), and note that  $P^{\Delta_r} h_0 = 0$  since  $h_0$  is a  $p$ th power, to obtain for  $r > 0$

$$\begin{aligned} (P^{\Delta_r} ** x) \quad h_0 P^{\Delta_r} P^{\Delta_0}(x) &= P^{\Delta_r}(h_1) P^{\Delta_{r_1}}(x) + \dots + P^{\Delta_r}(h_n) P^{\Delta_{r_n}}(x) \\ &\quad + h_1 P^{\Delta_r} P^{\Delta_{r_1}}(x) + \dots + h_n P^{\Delta_r} P^{\Delta_{r_n}}(x) \\ &= (P^{\Delta_r} h_1) P^{\Delta_{r_1}}(x) + \dots + P^{\Delta_r}(h_n) P^{\Delta_{r_n}}(x) \\ &\quad + h_1 P^{\Delta_{r_1}} P^{\Delta_r}(x) + \dots + h_n P^{\Delta_{r_n}} P^{\Delta_r}(x) \\ &= ((P^{\Delta_r} h_1) P^{\Delta_{r_1}} + \dots + (P^{\Delta_r} h_n) P^{\Delta_{r_n}})(x) \\ &\quad + (h_1 P^{\Delta_{r_1}} + \dots + h_n P^{\Delta_{r_n}})(P^{\Delta_r}(x)) \end{aligned}$$

$$\begin{aligned} \text{by } (*) \text{ applied} & \\ \text{to } P^{\Delta_r}(x) &= ((P^{\Delta_r} h_1) P^{\Delta_{r_1}} + \dots + (P^{\Delta_r} h_n) P^{\Delta_{r_n}})(x) \\ &\quad + h_0 P^{\Delta_0} P^{\Delta_r}(x), \end{aligned}$$

so rearranging gives

$$h_0 P^{\Delta_r}(x) = h_0 [P^{\Delta_r}, P^{\Delta_0}](x) = ((P^{\Delta_r} h_1) P^{\Delta_{r_1}} + \dots + (P^{\Delta_r} h_n) P^{\Delta_{r_n}})(x)$$

which yields the required result since  $x$  is arbitrary.  $\square$

The following lemma is the result of the relaxing effect of Appenzellerland on one of the authors.

LEMMA 1.4. *Suppose  $A^* \in UnAl/\mathcal{Q}^*$  contains at most finitely many algebraically independent elements. Let  $n$  be as in (1.1) and choose  $d_0, \dots, d_n \in A^*$ , with  $d_n$  a  $p$ th power, such that  $d_n P^{\Delta_0} + \dots + d_0 P^{\Delta_n} = 0 \in \text{End}(A^*)$ . Then*

$$P^{\Delta_i} d_j = \begin{cases} -d_n & i + j = n, 0 \leq j \leq n, \\ 0 & \text{otherwise}, 1 \leq i \leq n. \end{cases}$$

(N.B. By (1.1) none of the coefficients  $d_0, \dots, d_n$  can vanish.)

PROOF. From (1.3) we get

$$-d_n P^{\Delta_i} = (P^{\Delta_i} d_{n-1}) P^{\Delta_1} + \dots + (P^{\Delta_i} d_0) P^{\Delta_n}.$$

Rearranging then gives

$$0 = (P^{\Delta_i} d_{n-1}) P^{\Delta_1} + \dots + (P^{\Delta_i} d_{n-i} + d_n) P^{\Delta_i} + \dots + (P^{\Delta_i} d_0) P^{\Delta_n}.$$

But  $P^{\Delta_1}, \dots, P^{\Delta_n}$  are linearly independent by hypothesis, so

$$P^{\Delta_i} d_{n-j} = \begin{cases} 0 & i \neq j, \\ -d_n & i = j, \end{cases}$$

as required.  $\square$

THEOREM 1.5 (DICKSON).  $D^*(n) \simeq \mathbf{Z}/p[Y_1, \dots, Y_n]$  where  $\text{deg } Y_i = 2p^n - 2p^{n-i}$  and moreover

$$X^{p^n} + Y_1 X^{p^{n-1}} + \dots + Y_n X = \prod_{v \in V} (X - v)$$

in  $S^*[X]$ .

PROOF. First of all note that  $P^{\Delta_1}, \dots, P^{\Delta_n}$  are linearly independent on  $S^*$ . To see this, choose a basis  $v_1, \dots, v_n$  for  $V$  and consider

$$\text{Det}(P^{\Delta_i} v_j) = \begin{bmatrix} v_1^p, \dots, v_n^p \\ \vdots & & \vdots \\ v_1^{p^n}, \dots, v_n^{p^n} \end{bmatrix}.$$

Note that upon expanding the coefficient of the monomial  $v_1^p v_2^{p^2} \dots v_n^{p^n}$  is 1, so  $\text{Det}(P^{\Delta_i} v_j) \neq 0$ , and therefore the integer of Proposition 1.1 is in fact  $n$ , the

dimension of  $V$ . So from Lemmas (1.2) and (1.4) we find

$$P^{\Delta_i}(Y_k) = P^{\Delta_i} \frac{d_k}{d_0} = \frac{d_0 P^{\Delta_i} d_k - d_k P^{\Delta_i} d_0}{d_0^2}$$

$$= \begin{cases} \frac{-d_0 d_n}{d_0^2} = \frac{-d_n}{d_0} = -Y_n: & i = n - k; k > 0, \\ \frac{d_k d_n}{d_0^2} = Y_k Y_n: & i = n, \\ 0: & \text{otherwise,} \end{cases}$$

and thus

$$P^{\Delta_i} Y_k = \begin{cases} -Y_n: & i = n - k, \\ Y_k Y_n: & i = n, \\ 0: & \text{otherwise.} \end{cases}$$

Therefore

$$\text{Det}(P^{\Delta_i} Y_k) = \text{Det} \begin{bmatrix} 0 & \cdots & -Y_n & 0 \\ -Y_n & \cdots & & \vdots \\ Y_1 Y_n & \cdots & Y_{n-1} Y_n & Y_n Y_n \end{bmatrix} = (-1)^{n-1} Y_n^{n+1} \neq 0.$$

Therefore by the derivation lemma  $Y_1, \dots, Y_n \in S^*$  are algebraically independent.

Let  $H^* := \mathbf{Z}/p[Y_1, \dots, Y_n] < D^*(n) < S^*$ . Let  $F(\ )$  be the field of fractions functor. Note that  $F(S^*)$  is the field of rational functions on  $V$ , and that the elements of  $V$  all satisfy the equation  $f(X) = 0$  where

$$f(X) = \prod_{v \in V} (X - v) = Y_n X + \cdots + X^{p^n} \in H^*[X].$$

Thus we conclude that  $F(S^*)$  is in fact the splitting field of the polynomial  $f(X)$  over  $F(H^*)$ . Since the polynomial  $f(X)$  defines an additive function upon evaluating at elements of degree 2, it follows that the Galois group of  $F(S^*)$  over  $F(H^*)$  is a subgroup of  $GL(V)$ . (N.B. The extension is Galois since  $f(X)$  is separable.) Thus  $D^*(n) \leq S^* \cap F(H^*)$ . Since  $S^*$  is integral over  $H^*$  and  $H^*$  is integrally closed (in its field of fractions) it follows that  $S \cap F(H^*) = H^*$ . Thus  $D^*(n) \leq H^* \leq D^*(n)$  and the result follows.  $\square$

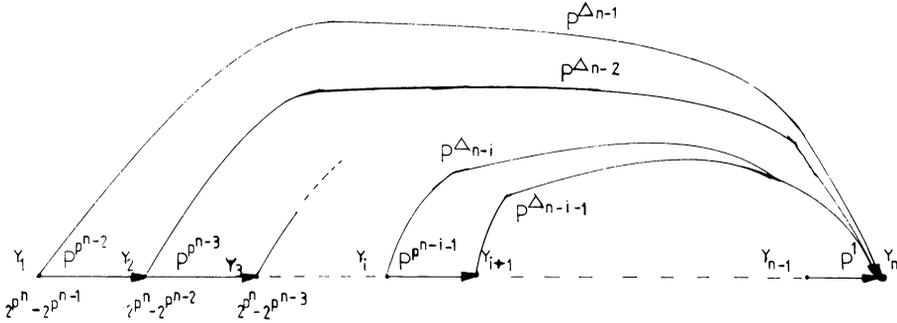
**COROLLARY TO THE PROOF OF (1.5).** *In the Dickson algebra  $D^*(n)$  we have*

$$P^{\Delta_i} Y_k = \left\{ \begin{array}{ll} -Y_n: & i + k = n, i \neq n, \\ Y_k Y_n: & i = n, \\ 0: & \text{otherwise,} \end{array} \right\} \quad 1 \leq k \leq n. \quad \square$$

**COROLLARY 1.6.** *In the Dickson algebra  $D^*(n)$  we have*

$$P^{p^j} Y_k = \left\{ \begin{array}{ll} Y_{k+1}: & \text{if } j + k = n - 1, \\ -Y_1 Y_k: & \text{if } j = n - 1, k > 1, \\ 0: & \text{otherwise.} \end{array} \right.$$

In particular the action of the Steenrod algebra on the module of indecomposables  $QD^*(n)$  is given by the following scheme.



PROOF. Note that as

$$\deg Y_i = 2p^n - 2p^{n-i},$$

the degree of  $Y_i$  increases with  $i$ . So  $Y_1$  has the lowest degree and  $Y_n$  the largest. Note further that

$$\deg Y_1^2 - \deg Y_n = 4p^n - 4p^{n-1} - 2p^n + 2 = 2p^{n-1}(p - 2) + 2 > 0.$$

Thus in the degrees  $2p^n - 2p^{n-i}$  there are no decomposable elements, and hence

$$P^{p^{n-i-1}}Y_i = aY_{i+1}; \quad a \in \mathbf{Z}/p.$$

To compute  $a$ , we apply the corollary to (1.5) to get

$$P^{\Delta_{n-i-1}}P^{p^{n-i-1}}Y_i = -aY_n \quad \text{and} \quad P^{\Delta_{n-i-1}}Y_i = 0.$$

So by the commutation rule and the corollary to (1.5) again we get

$$-Y_n = P^{\Delta_{n-i}}Y_i = [P^{\Delta_{n-i-1}}, P^{p^{n-i-1}}](Y_i) = -aY_n$$

giving  $a = 1$ , that is

$$P^{p^{n-i-1}}Y_i = Y_{i+1}.$$

Next consider the case  $P^{p^{n-1}}Y_k$ . For degree reasons we must have

$$(*) \quad P^{p^{n-1}}Y_k = aY_1Y_k; \quad a \in \mathbf{Z}/p.$$

To see this, note that

$$\begin{aligned} \deg P^{p^{n-1}}Y_k &= 2p^{n-1}(p - 1) + 2p^n - 2p^{n-k} = 2p^n - 2p^{n-1} + 2p^n - 2p^{n-k} \\ &= \deg Y_1 + \deg Y_k, \end{aligned}$$

and  $D(n)^*$  in this degree is one dimensional over  $\mathbf{Z}/p$ . To compute  $a \in \mathbf{Z}/p$  we apply  $P^{\Delta_{n-1}}$  to (\*) giving by the corollary to (1.5):

$$\begin{aligned} (P^{\Delta_{n-1}}(*)) \quad P^{\Delta_{n-1}}P^{p^{n-1}}Y_k &= a[(P^{\Delta_{n-1}}Y_1)Y_k + Y_1(P^{\Delta_{n-1}}Y_k)] \\ &= a[-Y_nY_k - \delta_{1,k}Y_kY_n] \\ &= -a(1 + \delta_{1,k})Y_kY_n. \end{aligned}$$

On the other hand

$$P^{\Delta_{n-1}}Y_k = \begin{cases} 0: & k \neq 1, \\ -Y_n: & k = 1. \end{cases}$$

Therefore for  $k \neq 1$  we have

$$Y_k Y_n = P^{\Delta_n} Y_k = [P^{\Delta_{n-1}}, P^{p^{n-1}}](Y_k) = -a Y_k Y_n + 0 = -a Y_k Y_n$$

and therefore  $a = -1$ , that is

$$P^{p^{n-1}} Y_k = -Y_1 Y_k: \quad k \neq 1.$$

Finally for  $k = 1$  we get by applying the corollary to (1.5),  $P^{\Delta_{n-1}}(*)$  for  $k = 1$ , and the preceding for  $k = n$ , that

$$\begin{aligned} Y_1 Y_n &= P^{\Delta_n} Y_1 = [P^{p^{n-1}}, P^{\Delta_{n-1}}](Y_1) \\ &= (2a) Y_1 Y_n - P^{p^{n-1}} Y_n = (1 + 2a) Y_1 Y_n \end{aligned}$$

so  $a = 0$  and  $P^{p^{n-1}} Y_1 = 0$  as required.

Summarizing we have shown

$$P^{p^j}(Y_k) = \begin{cases} Y_{k+1}: & \text{if } j = n - k - 1, \\ -Y_1 Y_k: & \text{if } j = n - 1, k > 1, \\ 0: & \text{if } j = n - 1, k = 1, \end{cases}$$

and so it remains to consider  $P^{p^j}(Y_k)$  for  $j \neq n - 1, n - k - 1$ . First of all note that

$$\deg Y_k = 2p^n - 2p^{n-k} < 2p^n$$

so

$$P^{p^j} Y_k = 0: \quad j \geq n.$$

Thus we need only consider values of  $j \leq n - 2$ . Next note, under the assumption  $j \leq n - 2$  that

$$\deg P^{p^j} Y_k < \deg P^{p^{n-1}} Y_n = 2p^{n-1}(p - 1) + 2p^n - 2 = \deg(Y_1 Y_n).$$

Finally note that through dimension  $2p^{n-1}(p - 1) + 2p^n - 2$  inclusive an additive basis for  $D^*(n)$  is given by the monomials

$$Y_1, \dots, Y_n, Y_1^2, Y_1 Y_2, \dots, Y_1 Y_n,$$

as a little bookkeeping shows. Further degree calculations give (again for  $j \leq n - 1$ )

$$\begin{aligned} \deg P^{p^j} Y_i &= \begin{cases} 2(p - 1)[p^{n-1} + \dots + 2p^j + \dots + p^{n-i}]: & j \geq n - i, \\ 2(p - 1)[p^{n-1} + \dots + p^{n-i} + p^j]: & j < n - i, \end{cases} \\ \deg Y_r &= 2(p - 1)[p^{n-1} + \dots + p^{n-r}], \\ \deg Y_1 Y_s &= 2(p - 1)[2p^{n-1} + p^{n-2} + \dots + p^{n-s}]. \end{aligned}$$

Thus the uniqueness of the  $p$ -adic expansion shows that  $P^{p^j} Y_i$  for  $i \leq n - 1$  lands in a nonzero degree iff  $j = n - i - 1$  (when  $r = i + 1$ ) or  $j = n - 1$  (when  $s = i$ ) as was to be shown.  $\square$

REMARK. It is not hard to show that Corollary 1.6 characterizes  $D^*(n)$  as an unstable algebra over the Steenrod algebra (see [15]).

**2. Nonrealization results.** With the explicit formulas of the preceding section for the action of  $P^{p^k}$  on the generators of the Dickson algebra, it is now an easy matter to prove

THEOREM 2.1. *Let  $p$  be a prime and  $n$  a positive integer. If there exists a topological space  $X$  such that  $H^*(X; \mathbf{Z}/p) \simeq D^*(n)$ , then either  $n = 1$  or  $n = 2$  and  $p \leq 3$ .*

PROOF. Consider first the case of  $p$  odd. If  $n > 2$  from (1.6) we see  $P^p Y_{n-2} = Y_{n-1}$ . Assume that there exists a space  $X$  such that  $H^*(X; \mathbf{Z}/p) \simeq D^*(n)$ . Then we can apply [10] to write

$$Y_{n-1} = P^p Y_{n-2} = \beta \Lambda Y_{n-2} + P^{p-2} \mathfrak{R} Y_{n-2}$$

where  $\Lambda$  and  $\mathfrak{R}$  are certain secondary operations of degrees  $2p(p-1)-1$  and  $4(p-1)$  respectively. For degree reasons one sees that

$$\Lambda Y_{n-2} = 0 = \mathfrak{R} Y_{n-2},$$

which leads to the contradiction that  $Y_{n-1} = 0$ . For  $n = 2$  we use the formulae

$$P^p Y_2 = -Y_1 Y_2, \quad P^1 Y_2 = 0$$

obtained from (1.6). Assuming  $H^*(X; \mathbf{Z}/p) \simeq D^*(2)$  we conclude (since  $\beta = 0$  on  $D^*(2)$ )

$$-Y_1 Y_2 = P^p Y_2 = P^{p-2} \mathfrak{R} Y_2$$

and hence  $\mathfrak{R} Y_2 \neq 0$ . Note

$$\deg Y_2 \leq \deg \mathfrak{R} Y_2 < \deg Y_1 Y_2,$$

so the only chance for  $\mathfrak{R} Y_2$  to be nonzero is

$$\mathfrak{R} Y_2 = b Y_1^2: \quad b \neq 0 \in \mathbf{Z}/p$$

which gives upon taking degrees

$$4(p-1) + 2p^2 - 2 = 2(2p^2 - 2p),$$

whose only solution is  $p = 3$ .

For  $p = 2$  note that the Steenrod algebra action is given by

$$\text{Sq}^{2^{i+1}} Y_j = \begin{cases} Y_{j+1}: & i+j = n-1, \\ Y_1 Y_n: & j = n-1, \\ 0: & \text{otherwise.} \end{cases}$$

So for  $n \geq 5$  we find,

$$\text{Sq}^{16} Y_{n-4} = Y_{n-3}$$

whereas

$$\text{Sq}^{2^i} Y_{n-4} = 0, \quad i = 1, 2, 3.$$

Applying the decomposition formula for  $Sq^{16}$  and reasoning as before eliminates  $D^*(n)$  for  $n \geq 5$  as a cohomology ring. The cases  $n = 3$  and  $4$  are eliminated by [7, 1.2].  $\square$

Combining (2.1) with [7, 18, p.73 and 10, 4.3] we arrive at the following result.

**THEOREM 2.2.** *A necessary and sufficient condition that  $D^*(n)$  be the  $\mathbf{Z}/p$  cohomology of a space is that either  $n = 1$ , or  $n = 2$  and  $p \leq 3$ .  $\square$*

**3. The (odd) prime 2.** For  $p = 2$  the Steenrod algebra acts unstably on the graded polynomial algebra  $P(W)$ , where the generators  $w \in W$  all have degree 1, the action being given by

$$Sq^i(w) = \begin{cases} w: & i = 0, \\ w^2: & i = 1, \\ 0: & i > 1. \end{cases}$$

The algebra  $\tilde{D}^*(n) := P(W)^{GL(W)}$  is then an unstable algebra over the Steenrod algebra. The structure of these Dickson algebras is given by the following result, analogous to those of §1.

**THEOREM 3.1.**  $\tilde{D}^*(n) \simeq \mathbf{Z}/2[Z_1, \dots, Z_n]$  where  $\deg Z_i = 2^n - 2^{n-i}$ . The action of the Steenrod algebra is given by

$$Sq^{\Delta_i} Z_j = \begin{cases} Z_n: & i + j = n, \\ 0: & \text{otherwise,} \end{cases}$$

where  $Sq^{\Delta_i} \in \mathcal{Q}^{2^i-1}(2)$  is the primitive element (usually denoted  $Q_i$ ) and

$$Sq^{2^i} Z_j = \begin{cases} Z_{j+1}: & i + j = n - 1, \\ Z_1 Z_j: & i = n - 1, \\ 0: & \text{otherwise.} \end{cases}$$

**COROLLARY 3.2.** *For  $n \geq 6$ ,  $\tilde{D}^*(n)$  is not the  $\mathbf{Z}/2$  cohomology of any space.*

**PROOF.** For  $n \geq 6$  one has

$$Sq^{16} Z_{n-5} = Z_{n-4},$$

whereas

$$Sq^{2^i} Z_{n-5} = 0: \quad i = 1, 2, 3,$$

and therefore as in the case of  $D^*(n)$  one sees that  $\tilde{D}^*(n)$  cannot be a cohomology ring.  $\square$

The algebras  $\tilde{D}^*(1)$ ,  $\tilde{D}^*(2)$  and  $\tilde{D}^*(3)$  occur as the  $\mathbf{Z}/2$  cohomology rings of the spaces  $\mathbf{R}P(\infty)$ ,  $BSO(3)$  and  $BG_2$  respectively. The remaining two cases,  $\tilde{D}^*(4)$  and  $\tilde{D}^*(5)$ , have type (8, 12, 14, 15) and (16, 24, 28, 30, 31), and it is an unsettled problem if they can occur as cohomology rings.

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